MATH1550, Winter 2023:
Exercise Set 6

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1. Let X and Y be discrete random variables.

y \ x	2	3	4
1	0.06	0.15	0.10
2	0.14	0.35	0.21

Determine whether this table corresponds to a valid joint probability distribution.

Solution. Note that for all pairs (x, y) where $x \in \{2, 3, 4\}$ and $y \in \{1, 2\}$, the joint probability at (x, y) is non-negative. Next we check to see if probabilities add up to 1:

$$0.06 + 0.15 + 0.10 + 0.14 + 0.35 + 0.21 = 1.01 \neq 1.$$

From this we conclude that the given table is not a valid joint probability distribution. □

2. Let X and Y be discrete random variables with joint probability distribution given by the following table:

y \ x	-3	2	4
1	0.1	k	0.2
3	0.3	0.1	0.1

(a) Determine the appropriate value for $k \in \mathbb{R}$ so that is is a valid joint probability distribution.

(b) Find the following probabilities
- $P(X = 2, Y = 3)$
- $P(X \leq 2, Y = 1)$
- $P(X < 2, Y = 1)$
- $P(X > 3, Y \leq 3)$
- $P(X = 2)$
- $P(Y \leq 3)$

Solution. (a) For the given table to be a valid joint probability distribution, the values in the table must add up to 1. Therefore, $k = 1 - (0.1 + 0.2 + 0.3 + 0.1 + 0.1) = 0.2$.

(b) Since $k = 0.2$ by part a), we have the following joint probability distribution:

y \ x	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Then,

- $P(X = 2, Y = 3) = 0.1$
- $P(X \leq 2, Y = 1) = P(X = -3, Y = 1) + P(X = 2, Y = 1) = 0.1 + 0.2 = 0.3$
- $P(X < 2, Y = 1) = P(X = -3, Y = 1) = 0.1$
- $P(X > 3, Y \leq 3) = P(X = 4, Y = 1) + P(X = 4, Y = 3) = 0.2 + 0.1 = 0.3$
- $P(X = 2) = P(X = 2, Y = 1) + P(X = 2, Y = 3) = 0.2 + 0.1 = 0.3$
- $P(Y \leq 3) = P(Y = 1) + P(Y = 3) = \big(P(X = -3, Y = 1) + P(X = 2, Y = 1) + P(X = 4, Y = 1)\big) + \big(P(X = -3, Y = 3) = P(X = 2, Y = 3) + P(X = 4, Y = 3)\big) = (0.1 + 0.2 + 0.2) + (0.3 + 0.1 + 0.1) = 1.$

□

3. A fair coin is tossed twice. Let X and Y be random variables such that

   - $X = 1$ if the first toss is heads, and $X = 0$ otherwise.
   - $Y = 1$ if both tosses are heads, and $Y = 0$ otherwise

   Give the joint probability distribution for X and Y.

Solution.

		x = 0	x = 1
y	0	0.5	0.25
	1	0	0.25

□

4. The joint probability density of continuous random variables X and Y is given by

$$f(x, y) = \begin{cases} \frac{2}{3}(x + 2y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Verify that this is a valid joint probability density function.

   (b) Find the following probabilities

   - $P(0 \leq X \leq 1, 0.5 \leq Y < 1)$
   - $P(0.25 \leq X \leq 0.5, 0 \leq Y < 1)$

Solution. (a) Observe that $f(x, y) \geq 0$ for all $x, y \in \mathbb{R}$. So, we only need to check $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dx\, dy \\
&= \int_0^1 \int_0^1 \frac{2}{3}(x + 2y)\, dx\, dy \qquad \text{(since } f(x, y) \text{ is zero outside of } 0 < x < 1, 0 < y < 1) \\
&= \int_0^1 \frac{2}{3}\left(\frac{x^2}{2} + 2yx\right)\Big|_0^1\, dy \\
&= \int_0^1 \frac{2}{3}\left(\frac{1}{2} + 2y\right) - 0\, dy \\
&= \frac{2}{3}\left(\frac{1}{2}y + y^2\right)\Big|_0^1 \\
&= \frac{2}{3}\left[\left(\frac{1}{2} + 1\right) - 0\right] \\
&= 1
\end{aligned}$$

Hence, the given function is a valid joint probability density function.

(b)

$$\begin{aligned}
P(0 \leq X \leq 1, 0.5 \leq Y < 1) &= \int_{0.5}^1 \int_0^1 \frac{2}{3}(x + 2y)\, dx\, dy \\
&= \int_{0.5}^1 \frac{2}{3}\left(\frac{x^2}{2} + 2yx\right)\Big|_0^1\, dy \\
&= \int_{0.5}^1 \frac{2}{3}\left(\frac{1}{2} + 2y\right) - 0\, dy \\
&= \frac{2}{3}\left(\frac{1}{2}y + y^2\right)\Big|_{0.5}^1 \\
&= \frac{2}{3}\left[\left(\frac{1}{2} + 1\right) - \left(\frac{1}{4} + \frac{1}{4}\right)\right] \\
&= \frac{2}{3}
\end{aligned}$$

$$
\begin{aligned}
P(0.25 \le X \le 0.5, 0 \le Y < 1) &= \int_0^1 \int_{0.25}^{0.5} \frac{2}{3}(x+2y)\ dx\ dy \\
&= \int_0^1 \frac{2}{3}\left(\frac{x^2}{2} + 2yx\right)\Big|_{0.25}^{0.5}\ dy \\
&= \int_0^1 \frac{2}{3}\left[\left(\frac{1}{8} + y\right) - \left(\frac{1}{32} + \frac{y}{2}\right)\right] dy \\
&= \frac{2}{3}\left[\left(\frac{1}{8}y + \frac{y^2}{2}\right) - \left(\frac{1}{32}y + \frac{y^2}{4}\right)\right]\Big|_0^1 \\
&= \frac{2}{3}\left[\left(\frac{1}{8} + \frac{1}{2}\right) - \left(\frac{1}{32} + \frac{1}{4}\right)\right] - 0 \\
&= \frac{2}{3} \cdot \left(\frac{5}{8} - \frac{9}{32}\right) \\
&= \frac{2}{3} \cdot \frac{11}{32} \\
&= \frac{11}{48}
\end{aligned}
$$

□

5. Let X and Y be continuous random variables defined on a joint sample space. Consider the function

$$
f(x,y) = \begin{cases} 2(x+4y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}
$$

Show that is is not a valid joint probability density of continuous random variables X and Y. Find an appropriate constant scaling factor to "salvage" this function.

Solution. Observe that $f(x,y) \ge 0$ for all $x, y \in \mathbb{R}$. So, we only need to check $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dx\ dy = 1$.

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dx\ dy \\
&= \int_0^1 \int_0^1 2(x+4y)\ dx\ dy \qquad \text{(since } f(x,y) \text{ is zero outside of } 0 < x < 1, 0 < y < 1) \\
&= \int_0^1 x^2 + 8yx\Big|_0^1\ dy \\
&= \int_0^1 (1+8y) - 0\ dy \\
&= y + 4y^2\Big|_0^1 \\
&= (1+4) - 0 \\
&= 5 \ne 1
\end{aligned}
$$

Hence, the given function is not a valid joint probability density function.

Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dx\ dy = 5$, scaling $f(x,y)$ by the factor $\frac{1}{5}$ (so that the integral equals 1) will salvage the function as a joint probability density. This gives the following joint probability density:

$$g(x,y) = \begin{cases} \frac{2}{5}(x+4y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

□

6. The joint probability density of continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} x+y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Determine the joint cumulative distribution function, and find $P(X < \frac{1}{2}, Y < 1)$.

Solution. The joint cumulative distribution function is defined as

$$F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\; ds\; dt.$$

If either $x \leq 0$ or $y \leq 0$ we have $f(x,y) = 0$, thus $F(x,y) = 0$.
For $0 < x, y < 1$ we have

$$\begin{aligned} F(x,y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\; ds\; dt \\ &= \int_{0}^{y} \int_{0}^{x} s + t\; ds\; dt \\ &= \int_{0}^{y} \frac{s^2}{2} + st \Big|_0^x \; dt \\ &= \int_{0}^{y} \frac{x^2}{2} + xt\; dt \\ &= \frac{x^2 t}{2} + \frac{x t^2}{2} \Big|_0^y \\ &= \frac{x^2 y}{2} + \frac{x y^2}{2} \end{aligned}$$

For $0 < x < 1$, $y \geq 1$ we have

$$\begin{aligned} F(x,y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\; ds\; dt \\ &= \int_{0}^{1} \int_{0}^{x} s + t\; ds\; dt \\ &= \int_{0}^{1} \frac{s^2}{2} + st \Big|_0^x \; dt \\ &= \int_{0}^{1} \frac{x^2}{2} + xt\; dt \\ &= \frac{x^2 t}{2} + \frac{x t^2}{2} \Big|_0^1 \\ &= \frac{x^2}{2} + \frac{x}{2} \end{aligned}$$

For $0 < y < 1$, $x \geq 1$ we have

$$\begin{aligned} F(x,y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\ ds\ dt \\ &= \int_{0}^{y} \int_{0}^{1} s + t\ ds\ dt \\ &= \int_{0}^{y} \frac{s^2}{2} + st\Big|_0^1\ dt \\ &= \int_{0}^{y} \frac{1}{2} + t\ dt \\ &= \frac{t}{2} + \frac{t^2}{2}\Big|_0^y \\ &= \frac{y^2}{2} + \frac{y}{2} \end{aligned}$$

For $x \geq 1$ and $y \geq 1$ we have

$$\begin{aligned} F(x,y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\ ds\ dt \\ &= \int_{0}^{1} \int_{0}^{1} s + t\ ds\ dt \\ &= 1 \end{aligned}$$

In summary

$$F(x,y) = \begin{cases} 0 & \text{for } x \leq 0 \text{ or } y \leq 0 \\ \frac{x^2y}{2} + \frac{xy^2}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ \frac{x^2}{2} + \frac{x}{2} & \text{for } 0 < x < 1, y \geq 1 \\ \frac{y^2}{2} + \frac{y}{2} & \text{for } x \geq 1, 0 < y < 1 \\ 1 & \text{for } x \geq 1, y \geq 1 \end{cases}$$

Then

$$P\left(X < \frac{1}{2}, Y < 1\right) = F\left(\frac{1}{2}, 1\right) = \frac{(\frac{1}{2})^2}{2} + \frac{(\frac{1}{2})}{2} = \frac{3}{8}.$$

□

7. The joint probability density of continuous random variables X and Y is given by

$$f(x) = \begin{cases} \frac{2}{55}(x + 27) & \text{for } 0 \leq x \leq 1, 1 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Verify that this is a valid joint probability density function.
   (b) Find the following probabilities
      - $P(0 \leq X \leq 1, 1.5 \leq Y < 2)$
      - $P(0.25 \leq X \leq 0.5, 1 \leq Y < 2)$
      - $P(0.5 \leq X \leq 1, 1.25 \leq Y < 1.5)$
   (c) Find the joint cumulative distribution function.

Solution. (a) Observe that $f(x,y) \geq 0$ for all $x, y \in \mathbb{R}$. So, we only need to check $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dx\ dy = 1$.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dx\ dy \\
&= \int_{1}^{2}\int_{0}^{1} \frac{2}{55}(x+27)\ dx\ dy \qquad \text{(as } f(x,y) = 0 \text{ outside of } 0 < x < 1, 1 < y < 2\text{)} \\
&= \int_{1}^{2} \frac{2}{55}\left(\frac{x^2}{2} + 27x\right)\Big|_0^1\ dy \\
&= \int_{1}^{2} \frac{2}{55}\left(\frac{1}{2} + 27\right) - 0\ dy \\
&= \frac{2}{55}\left(\frac{1}{2}y + 27y\right)\Big|_1^2 \\
&= \frac{2}{55}\left[(1+54) - \left(\frac{1}{2} + 27\right)\right] \\
&= 1
\end{aligned}$$

Hence, the given function is a valid joint probability density function.

(b)

$$\begin{aligned}
P(0 \leq X \leq 1, 1.5 \leq Y < 2) &= \int_{1.5}^{2}\int_{0}^{1} \frac{2}{55}(x+27)\ dx\ dy \\
&= \int_{1.5}^{2} \frac{2}{55}\left(\frac{x^2}{2} + 27x\right)\Big|_0^1\ dy \\
&= \int_{1.5}^{2} \frac{2}{55}\left(\frac{1}{2} + 27\right) - 0\ dy \\
&= \frac{2}{55}\left(\frac{1}{2}y + 27y\right)\Big|_{1.5}^2 \\
&= \frac{2}{55}\left[(1+54) - \left(\frac{3}{4} + \frac{81}{2}\right)\right] \\
&= \frac{2}{55} \cdot \frac{55}{4} \\
&= \frac{1}{2}
\end{aligned}$$

$$
\begin{aligned}
P(0.25 \le X \le 0.5, 1 \le Y < 2) &= \int_1^2 \int_{0.25}^{0.5} \frac{2}{55}(x+27)\,dx\,dy \\
&= \int_1^2 \frac{2}{55}\left(\frac{x^2}{2}+27x\right)\Big|_{0.25}^{0.5}\,dy \\
&= \int_1^2 \frac{2}{55}\left[\left(\frac{1}{8}+\frac{27}{2}\right)-\left(\frac{1}{32}+\frac{27}{4}\right)\right]dy \\
&= \frac{2}{55}\left(\frac{109}{8}y-\frac{217}{32}y\right)\Big|_1^2 \\
&= \frac{2}{55}\cdot\frac{219}{32}y\Big|_1^2 \\
&= \frac{2}{55}\cdot\frac{219}{32}(2-1) \\
&= \frac{219}{880} \approx 0.25
\end{aligned}
$$

$$
\begin{aligned}
P(0.5 \le X \le 1, 1.25 \le Y < 1.5) &= \int_{1.25}^{1.5} \int_{0.5}^{1} \frac{2}{55}(x+27)\,dx\,dy \\
&= \int_{1.25}^{1.5} \frac{2}{55}\left(\frac{x^2}{2}+27x\right)\Big|_{0.5}^{1}\,dy \\
&= \int_{1.25}^{1.5} \frac{2}{55}\left[\left(\frac{1}{2}+27\right)-\left(\frac{1}{8}+\frac{27}{2}\right)\right]dy \\
&= \frac{2}{55}\cdot\frac{111}{8}y\Big|_{1.25}^{1.5} \\
&= \frac{2}{55}\cdot\frac{111}{8}\cdot(1.5-1.25) \\
&= \frac{111}{880} \approx 0.13
\end{aligned}
$$

(c) The joint cumulative distribution function is defined as

$$
F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt.
$$

If either $x < 0$ or $y \le 1$ we have $f(x,y) = 0$, thus $F(x,y) = 0$.
For $0 \le x \le 1, 1 < y < 2$:

$$
\begin{aligned}
F(x,y) &= \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt \\
&= \int_1^y \int_0^x \frac{2}{55}(s+27)\,ds\,dt \\
&= \int_1^y \frac{2}{55}\left(\frac{s^2}{2}+27s\right)\Big|_0^x\,dt \\
&= \int_1^y \frac{2}{55}\left(\frac{x^2}{2}+27x\right)-0\,dt \\
&= \frac{2}{55}\left(\frac{x^2}{2}+27x\right)t\Big|_1^y \\
&= \frac{2}{55}\left[\left(\frac{x^2}{2}+27x\right)y-\left(\frac{x^2}{2}+27x\right)\right]
\end{aligned}
$$

For $0 \le x \le 1$, $y \ge 2$:

$$\begin{aligned}
F(x,y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\ ds\ dt \\
&= \int_{1}^{2} \int_{0}^{x} \frac{2}{55}(s+27)\ ds\ dt \quad \text{(as } y \ge 2 \text{ and } f(x,y) = 0 \text{ outside of } 0 < x < 1,\ 1 < y < 2\text{)} \\
&= \int_{1}^{2} \frac{2}{55} \left(\frac{s^2}{2} + 27s \right) \Big|_0^x\ dt \\
&= \int_{1}^{2} \frac{2}{55} \left(\frac{x^2}{2} + 27x \right) - 0\ dt \\
&= \frac{2}{55} \left(\frac{x^2}{2} + 27x \right) t \Big|_1^2 \\
&= \frac{2}{55} \left(\frac{x^2}{2} + 27x \right) (2-1) \\
&= \frac{2}{55} \left(\frac{x^2}{2} + 27x \right)
\end{aligned}$$

For $1 < y < 2$, $x > 1$:

$$\begin{aligned}
F(x,y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\ ds\ dt \\
&= \int_{1}^{y} \int_{0}^{1} \frac{2}{55}(s+27)\ ds\ dt \quad \text{(as } x > 1 \text{ and } f(x,y) = 0 \text{ outside of } 0 < x < 1,\ 1 < y < 2\text{)} \\
&= \int_{1}^{y} \frac{2}{55} \left(\frac{s^2}{2} + 27s \right) \Big|_0^1\ dt \\
&= \int_{1}^{y} \frac{2}{55} \left(\frac{1}{2} + 27 \right) - 0\ dt \\
&= \frac{2}{55} \cdot \frac{55}{2} t \Big|_1^y \\
&= y - 1
\end{aligned}$$

For $x > 1$ and $y \ge 2$:

$$\begin{aligned}
F(x,y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f(s,t)\ ds\ dt \\
&= \int_{1}^{2} \int_{0}^{1} \frac{2}{55}(s+27)\ ds\ dt \\
&= 1
\end{aligned}$$

In summary

$$F(x,y) = \begin{cases}
0 & \text{for } x < 0 \text{ or } y \le 1 \\
\frac{2}{55}\left[\left(\frac{x^2}{2} + 27x \right) y - \left(\frac{x^2}{2} + 27x \right) \right] & \text{for } 0 \le x \le 1, 1 < y < 2 \\
\frac{2}{55} \left(\frac{x^2}{2} + 27x \right) & \text{for } 0 \le x \le 1, y \ge 2 \\
y - 1 & \text{for } x > 1, 1 < y < 2 \\
1 & \text{for } x > 1, y \ge 2
\end{cases}$$

□

8. The joint probability density of continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Verify that this is a valid joint probability density function.

(b) Find the joint cumulative distribution function.

(c) Use part (b) to find

- $P(X \le 1, Y \le 0.5)$
- $P(0.25 < X \le 0.5, Y \le 1)$
- $P(0.25 \le X \le 0.5, 0.5 < Y \le 1)$

Solution. (a) Observe that $f(x,y) \ge 0$ for all $x, y \in \mathbb{R}$. So, we only need to check $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \\
&= \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy \quad (\text{as } f(x,y) = 0 \text{ outside of } 0 < x < 1, 0 < y < 1) \\
&= \int_0^1 \frac{2}{5}(x^2+3yx)\Big|_0^1\,dy \\
&= \int_0^1 \frac{2}{5}(1+3y) - 0\,dy \\
&= \frac{2}{5}\left(y + \frac{3}{2}y^2\right)\Big|_0^1 \\
&= \frac{2}{5}\left(1+\frac{3}{2}\right) - 0 \\
&= 1
\end{aligned}$$

Hence, the given function is a valid joint probability density function.

(b) The joint cumulative distribution function is defined

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt.$$

If either $x \le 0$ or $y \le 0$ we have $f(x,y) = 0$, thus $F(x,y) = 0$.

For $0 < x, y < 1$:

$$\begin{aligned}
F(x,y) &= \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt \\
&= \int_0^y\int_0^x \frac{2}{5}(2s+3t)\,ds\,dt \\
&= \int_0^y \frac{2}{5}(s^2+3ts)\Big|_0^x\,dt \\
&= \int_0^y \frac{2}{5}(x^2+3tx) - 0\,dt \\
&= \frac{2}{5}\left(x^2t + \frac{3}{2}t^2x\right)\Big|_0^y \\
&= \frac{2}{5}\left(x^2y + \frac{3}{2}y^2x\right)
\end{aligned}$$

For $0 < x < 1$, $y \geq 1$:

$$\begin{aligned}
F(x,y) &= \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\ ds\ dt \\
&= \int_{0}^{1}\int_{0}^{x} \frac{2}{5}(2s+3t)\ ds\ dt \quad \text{(as } y \geq 1\text{, and } f(x,y)=0 \text{ outside of } 0<x<1, 0<y<1\text{)} \\
&= \int_{0}^{1} \frac{2}{5}(s^2+3ts)\Big|_0^x\ dt \\
&= \int_{0}^{1} \frac{2}{5}(x^2+3tx) - 0\ dt \\
&= \frac{2}{5}\left(x^2 t + \frac{3}{2}t^2 x\right)\Big|_0^1 \\
&= \frac{2}{5}\left(x^2 + \frac{3}{2}x\right) - 0 \\
&= \frac{2}{5}\left(x^2 + \frac{3}{2}x\right)
\end{aligned}$$

For $0 < y < 1$, $x \geq 1$:

$$\begin{aligned}
F(x,y) &= \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\ ds\ dt \\
&= \int_{0}^{y}\int_{0}^{1} \frac{2}{5}(2s+3t)\ ds\ dt \quad \text{(as } x \geq 1\text{, and } f(x,y)=0 \text{ outside of } 0<x<1, 0<y<1\text{)} \\
&= \int_{0}^{y} \frac{2}{5}(s^2+3ts)\Big|_0^1\ dt \\
&= \int_{0}^{y} \frac{2}{5}(1+3t) - 0\ dt \\
&= \frac{2}{5}\left(t + \frac{3}{2}t^2\right)\Big|_0^y \\
&= \frac{2}{5}\left(y + \frac{3}{2}y^2\right) - 0 \\
&= \frac{2}{5}\left(y + \frac{3}{2}y^2\right)
\end{aligned}$$

For $x \geq 1$ and $y \geq 1$ we have

$$\begin{aligned}
F(x,y) &= \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\ ds\ dt \\
&= \int_{0}^{1}\int_{0}^{1} \frac{2}{5}(2s+3t)\ ds\ dt \\
&= 1
\end{aligned}$$

In summary

$$F(x,y) = \begin{cases} 0 & \text{for } x \le 0 \text{ or } y \le 0 \\ \frac{2}{5}\left(x^2 y + \dfrac{3}{2} y^2 x\right) & \text{for } 0 < x < 1, 0 < y < 1 \\ \frac{2}{5}\left(x^2 + \dfrac{3}{2} x\right) & \text{for } 0 < x < 1, y \ge 1 \\ \frac{2}{5}\left(y + \dfrac{3}{2} y^2\right) & \text{for } x \ge 1, 0 < y < 1 \\ 1 & \text{for } x \ge 1, y \ge 1 \end{cases}$$

(c) Using the joint cumulative distribution function,

$$P(X \le 1, Y \le 0.5) = F(1, 0.5).$$

This falls into the fourth part of $F(x,y)$ that we have found in (b). So,

$$P(X \le 1, Y \le 0.5) = F(1, 0.5) = \frac{2}{5}\left(0.5 + \frac{3}{2}(0.5)^2\right) = \frac{14}{40} = \frac{7}{20} = 0.35.$$

Next we have

$$\begin{aligned} P(0.25 < X \le 0.5, Y \le 1) &= P(X \le 0.5, Y \le 1) - P(X \le 0.25, Y \le 1) \\ &= F(0.5, 1) - F(0.25, 1). \end{aligned}$$

Both terms in the expression on the right can be found from the third part of $F(x,y)$. So,

$$\begin{aligned} P(0.25 < X \le 0.5, Y \le 1) &= F(0.5, 1) - F(0.25, 1) \\ &= \frac{2}{5}\left((0.5)^2 + \frac{3}{2} \cdot 0.5\right) - \frac{2}{5}\left((0.25)^2 + \frac{3}{2} \cdot 0.25\right) \\ &= 0.225 \end{aligned}$$

Finally,

$$P(0.25 \le X \le 0.5, 0.5 < Y \le 1) = P(0.25 \le X \le 0.5, Y \le 1) - P(0.25 < X \le 0.5, Y \le 0.5).$$

We have already found $P(0.25 < X \le 0.5, Y \le 1) = 0.225$, so it remains to find

$$\begin{aligned} P(0.25 < X \le 0.5, Y \le 0.5) &= P(X \le 0.5, Y \le 0.5) - P(X \le 0.25, Y \le 0.5) \\ &= F(0.5, 0.5) - F(0.25, 0.5). \end{aligned}$$

Both terms in the expression on the right can be found from the second part of $F(x,y)$. So,

$$\begin{aligned} & P(0.25 < X \le 0.5, Y \le 0.5) \\ &= P(X \le 0.5, Y \le 0.5) - P(X \le 0.25, Y \le 0.5) \\ &= F(0.5, 0.5) - F(0.25, 0.5) \\ &= \frac{2}{5}\left((0.5)^2 \cdot (0.5) + \frac{3}{2}(0.5)^2 \cdot (0.5)\right) - \frac{2}{5}\left((0.25)^2 \cdot (0.5) + \frac{3}{2}(0.5)^2 \cdot 0.25\right) \\ &= \frac{1}{8} - \frac{1}{20} \\ &= 0.075 \end{aligned}$$

Thus

$$\begin{aligned}
&P(0.25 \le X \le 0.5, 0.5 < Y \le 1) \\
&= P(0.25 \le X \le 0.5, Y \le 1) - P(0.25 < X \le 0.5, Y \le 0.5) \\
&= 0.225 - 0.075 \\
&= 0.15.
\end{aligned}$$

□

9. Let X and Y be discrete random variables with joint probability distribution given by the following table:

y \ x	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Find the marginal distributions for X and Y.

Solution. The marginal distribution $g(x)$ for X is given by

$$\begin{aligned}
g(-3) &= 0.1 + 0.3 = 0.4, \\
g(2) &= 0.2 + 0.1 = 0.3, \\
g(4) &= 0.2 + 0.1 = 0.3.
\end{aligned}$$

The marginal distribution $h(y)$ for Y is given by

$$\begin{aligned}
h(1) &= 0.1 + 0.2 + 0.2 = 0.5, \\
h(3) &= 0.3 + 0.1 + 0.1 = 0.5.
\end{aligned}$$

□

10. The joint distribution function, $f(x,y)$, for discrete random variables X and Y is given below. Find $F(3,3)$ where $F(x,y)$ is the cumulative distribution function for X and Y.

y \ x	1	2
-2	0.1	0.2
-1	0.2	0.1
4	0	0.1
5	0.3	0

Solution.

$$\begin{aligned} F(3,3) &= P(X \le 3, Y \le 3) \\ &= f(1,-2) + f(-1,-1) + f(2,-1) + f(2,-1) \\ &= 0.1 + 0.2 + 0.2 + 0.1 \\ &= 0.6. \end{aligned}$$

□

11. A fair coin is tossed 4 times. Let random variable X be the number of heads appearing in the 4 tosses and Y be the largest number of consecutive heads in the 4 tosses. If $f(x,y)$ is the joint probability distribution for X and Y, find $f(3,2)$. *(For practice find the entire joint distribution.)*

Solution. The joint distribution $f(x,y)$ is summarized in the table below.

$y \backslash x$	0	1	2	3	4
0	$\frac{1}{16}$				
1		$\frac{4}{16}$	$\frac{3}{16}$		
2			$\frac{3}{16}$	$\frac{2}{16}$	
3				$\frac{2}{16}$	
4					$\frac{1}{16}$

To find $f(3,2)$ for example:

$$f(3,2) = P(X = 3, Y = 2) = P(\{HHTH, HTHH\}) = \frac{2}{16}.$$

□

12. The joint probability density function for continuous random variables is given below. Find $P(0 \le X \le \frac{1}{2}, \frac{1}{2} \le Y \le 1)$.

$$f(x,y) = \begin{cases} 12xy(1-x) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Solution.

$$
\begin{aligned}
P\left(0 \le X \le \frac{1}{2}, \frac{1}{2} \le Y \le 1\right) &= \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} 12xy(1-x)\ dx\ dy \\
&= 12 \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{1}{2}} xy - x^2 y\ dx\ dy \\
&= 12 \int_{\frac{1}{2}}^{1} \left[\frac{x^2 y}{2} - \frac{x^3 y}{3}\right]_0^{\frac{1}{2}} dy \\
&= 12 \int_{\frac{1}{2}}^{1} \frac{y}{12}\ dy \\
&= \left.\frac{y^2}{2}\right|_{\frac{1}{2}}^{1} \\
&= \left(\frac{1}{2} - \frac{1}{8}\right) \\
&= \frac{3}{8}
\end{aligned}
$$

□

13. Is the following function a valid joint density function?

$$
f(x, y) = \begin{cases} \frac{x+y}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}
$$

Solution. No, since

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x, y)\ dx\ dy &= \int_0^1 \int_0^1 \frac{x+y}{2}\ dx\ dy \\
&= \int_0^1 \left[\frac{x^2}{4} + \frac{xy}{2}\right]_0^1 dy \\
&= \int_0^1 \frac{1}{4} + \frac{y}{2}\ dy \\
&= \left.\frac{y}{4} + \frac{y^2}{4}\right|_0^1 \\
&= \frac{1}{2} \\
&\ne 1
\end{aligned}
$$

□

14. The joint distribution, $f(x, y)$, for discrete random variables X and Y is given below. Let $g(x)$ be the marginal distribution for X. Find $g(4)$.

y \ x	1	2	3	4	5	6
2	$\frac{1}{36}$					
3		$\frac{2}{36}$				
4		$\frac{1}{36}$	$\frac{2}{36}$			
5			$\frac{2}{36}$	$\frac{2}{36}$		
6			$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	
7				$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
8				$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
9					$\frac{2}{36}$	$\frac{2}{36}$
10					$\frac{1}{36}$	$\frac{2}{36}$
11						$\frac{2}{36}$
12						$\frac{1}{36}$

Solution.

$$g(4) = \sum_{y=2}^{12} f(4, y) = \frac{2}{36} + \frac{2}{36} + \frac{2}{36} + \frac{1}{36} = \frac{7}{36}.$$

□

15. The joint probability density function for continuous random variables is given below. Find $g(x)$, the marginal density for X.

$$f(x, y) = \begin{cases} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution.

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x, y)\, dy \\ &= \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\, dy \\ &= \frac{6}{7}\left(x^2 y + \frac{xy^2}{4}\right)\Big|_0^2 \\ &= \frac{6}{7}\left(2x^2 + x\right) \end{aligned}$$

□