MATH1550, Winter 2023:
Exercise Set 6

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1. Let X and Y be discrete random variables.

		x		
		2	3	4
y	1	0.06	0.15	0.10
	2	0.14	0.35	0.21

   Determine whether this table corresponds to a valid joint probability distribution.

2. Let X and Y be discrete random variables with joint probability distribution given by the following table:

		x		
		-3	2	4
y	1	0.1	k	0.2
	3	0.3	0.1	0.1

   (a) Determine the appropriate value for $k \in \mathbb{R}$ so that is is a valid joint probability distribution.

   (b) Find the following probabilities
   - $P(X = 2, Y = 3)$
   - $P(X \leq 2, Y = 1)$
   - $P(X < 2, Y = 1)$
   - $P(X > 3, Y \leq 3)$
   - $P(X = 2)$
   - $P(Y \leq 3)$

3. A fair coin is tossed twice. Let X and Y be random variables such that
   - $X = 1$ if the first toss is heads, and $X = 0$ otherwise.
   - $Y = 1$ if both tosses are heads, and $Y = 0$ otherwise

   Give the joint probability distribution for X and Y.

4. The joint probability density of continuous random variables X and Y is given by

$$f(x, y) = \begin{cases} \frac{2}{3}(x + 2y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Verify that this is a valid joint probability density function.

   (b) Find the following probabilities
   - $P(0 \leq X \leq 1, 0.5 \leq Y < 1)$
   - $P(0.25 \leq X \leq 0.5, 0 \leq Y < 1)$

5. Let X and Y be continuous random variables defined on a joint sample space. Consider the function

$$f(x,y) = \begin{cases} 2(x+4y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Show that is is not a valid joint probability density of continuous random variables X and Y. Find an appropriate constant scaling factor to "salvage" this function.

6. The joint probability density of continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} x+y & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Determine the joint cumulative distribution function, and find $P(X < \frac{1}{2}, Y < 1)$.

7. The joint probability density of continuous random variables X and Y is given by

$$f(x) = \begin{cases} \frac{2}{55}(x+27) & \text{for } 0 \leq x \leq 1, 1 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Verify that this is a valid joint probability density function.

   (b) Find the following probabilities
   - $P(0 \leq X \leq 1, 1.5 \leq Y < 2)$
   - $P(0.25 \leq X \leq 0.5, 1 \leq Y < 2)$
   - $P(0.5 \leq X \leq 1, 1.25 \leq Y < 1.5)$

   (c) Find the joint cumulative distribution function.

8. The joint probability density of continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Verify that this is a valid joint probability density function.

   (b) Find the joint cumulative distribution function.

   (c) Use part (b) to find
   - $P(X \leq 1, Y \leq 0.5)$
   - $P(0.25 < X \leq 0.5, Y \leq 1)$
   - $P(0.25 \leq X \leq 0.5, 0.5 < Y \leq 1)$

9. Let X and Y be discrete random variables with joint probability distribution given by the following table:

		x		
		-3	2	4
y	1	0.1	0.2	0.2
	3	0.3	0.1	0.1

Find the marginal distributions for X and Y.

10. The joint distribution function, $f(x, y)$, for discrete random variables X and Y is given below. Find $F(3, 3)$ where $F(x, y)$ is the cumulative distribution function for X and Y.

		x	
		1	2
y	-2	0.1	0.2
	-1	0.2	0.1
	4	0	0.1
	5	0.3	0

11. A fair coin is tossed 4 times. Let random variable X be the number of heads appearing in the 4 tosses and Y be the largest number of consecutive heads in the 4 tosses. If $f(x, y)$ is the joint probability distribution for X and Y, find $f(3, 2)$. *(For practice find the entire joint distribution.)*

12. The joint probability density function for continuous random variables is given below. Find $P(0 \leq X \leq \frac{1}{2}, \frac{1}{2} \leq Y \leq 1)$.

$$f(x, y) = \begin{cases} 12xy(1-x) & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

13. Is the following function a valid joint density function?

$$f(x, y) = \begin{cases} \frac{x+y}{2} & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

14. The joint distribution, $f(x, y)$, for discrete random variables X and Y is given below. Let $g(x)$ be the marginal distribution for X. Find $g(4)$.

		x					
		1	2	3	4	5	6
	2	$\frac{1}{36}$					
	3		$\frac{2}{36}$				
	4		$\frac{1}{36}$	$\frac{2}{36}$			
	5			$\frac{2}{36}$	$\frac{2}{36}$		
	6			$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	
y	7				$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
	8				$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
	9					$\frac{2}{36}$	$\frac{2}{36}$
	10					$\frac{1}{36}$	$\frac{2}{36}$
	11						$\frac{2}{36}$
	12						$\frac{1}{36}$

15. The joint probability density function for continuous random variables is given below. Find $g(x)$, the marginal density for X.

$$f(x,y) = \begin{cases} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) & \text{for } 0 < x < 1, 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$