MATH1550, Winter 2023:
Exercise Set 5

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1. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{10}(3x^2+1) & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Verify that $f(x)$ is a valid probability density function

   (b) Find $P(X \ge 1)$

2. Let Y be a continuous random variable. Let $f(x) = k(1+x)$ for $x \in [0,2]$ and $f(x) = 0$ elsewhere. For which values of k is f a valid probability density function for Y?

3. Which of the following are allowable as probability density functions for some continuous random variable? (show why or why not)

   (a)
$$f(x) = \begin{cases} 4x^3 & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (b)
$$g(x) = \begin{cases} 6x^2 - 2x & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (c)
$$h(x) = \begin{cases} \frac{1}{6}(1+x)^5 & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

   (d)
$$p(x) = \begin{cases} \frac{3}{4}(1-x^2) & \text{for } -1 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

4. Find the cumulative distribution function $F(x)$ for the random variable X in question **??**, and use $F(x)$ to compute $P(-1 \le X \le 1)$ and $P(0.5 \le X \le 1.5)$.

5. Find a probability density function for the random variable whose cumulative distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x \le 0 \\ x & \text{for } 0 < x < 1 \\ 1 & \text{for } x \ge 1 \end{cases} .$$

6. The probability density function of a random variable X is given by

$$f(x) = \begin{cases} \frac{c}{\sqrt{x}} & \text{for } 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

   Find the value of c, and compute $P(X < \frac{1}{4})$ and $P(X > 1)$.

7. Suppose discrete random variable X has range $\{0, 1, 2\}$ with probability distribution

$$f(x) = \frac{\binom{2}{x}\binom{4}{3-x}}{\binom{6}{3}}.$$

   (a) Verify that this is a valid probability distribution.
   (b) Create a histogram for this probability distribution.
   (c) Give the cumulative probability distribution for X.
   (d) Come up with an example of a probability experiment which corresponds to this X.

8. Suppose the probability density of continuous random variable X is given by

$$f(x) = \begin{cases} 4x^3 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

   Find the cumulative distribution function $F(x)$ for X, and use it to compute $P(0.5 < X < 1)$.

9. Suppose the probability density of continuous random variable X is given by

$$f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 < x \leq 1 \\ \frac{1}{2} & \text{for } 1 < x \leq 2 \\ \frac{3-x}{2} & \text{for } 2 < x < 3 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find the cumulative distribution function $F(x)$ for X.

   (b) Use the cumulative distribution to compute the following probabilities
   - $P(0.25 < x < 0.5)$
   - $P(0.5 < x < 1.5)$
   - $P(0.5 < x < 2.25)$

10. The continuous random variable X has cumulative distribution function given by

$$F(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ \frac{x+1}{2} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

   (a) Compute the following probabilities
   - $P\left(-\frac{1}{2} < X < \frac{1}{2}\right)$
   - $P\left(2 < X < 3\right)$

   (b) Determine the probability density function for X.

11. Find the probability density function for continuous random variable Y with cumulative distribution function given by

$$F(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ \frac{1}{4}y^2 & \text{for } 0 \leq y \leq 2 \\ 1 & \text{for } y > 2 \end{cases}$$

12. Can the following function serve as a valid probability density for a continuous random variable?

$$f(x) = \begin{cases} \frac{2}{3}(x+1) & \text{for } x \in [0,1] \\ \\ 0 & \text{otherwise} \end{cases}$$

13. Can the following function serve as a valid probability density for a continuous random variable?

$$f(x) = \begin{cases} \frac{1}{4}(x+1) & \text{for } x \in [2,4] \\ \\ 0 & \text{otherwise} \end{cases}$$

14. Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} \frac{x+1}{8} & \text{for } x \in (2,4) \\ \\ 0 & \text{otherwise} \end{cases}$$

Find $P(1.5 < X < 3)$.

15. Determine the appropriate value for k so that the following function is a valid probability density

$$f(x) = \begin{cases} \frac{k}{\sqrt{x}} & \text{for } x \in (0,4] \\ \\ 0 & \text{otherwise} \end{cases}$$

16. The probability density for a continuous random variable X is given below. Find $P(X > \frac{1}{2})$.

$$f(x) = \begin{cases} 6x(1-x) & \text{for } x \in (0,1) \\ \\ 0 & \text{otherwise} \end{cases}$$

17. The probability density for a continuous random variable X is given below. Find $P(-0.5 < X \leq 0.25)$.

$$f(x) = \begin{cases} x+1 & \text{for } x \in [-1,0) \\ \\ 1-x & \text{for } x \in [0,1] \\ \\ 0 & \text{otherwise} \end{cases}$$

18. Let X be a continuous random variable with probability density given by

$$f(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 \le x \le 2 \\ \\ 0 & \text{otherwise} \end{cases}$$

Find the cumulative distribution function for X.

Fill in blank:

$$F(x) = \underline{\quad 0 \quad} \text{ for } x < 0$$

$$F(x) = \underline{\qquad} \text{ for } 0 \le x \le 2$$

$$F(x) = \underline{\quad 1 \quad} \text{ for } x > 2$$

19. The cumulative distribution function for a continuous random variable X is given below. Find $P(\frac{1}{4} \le X \le 1)$.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \\ \sin(\pi x) & \text{for } 0 \le x \le \frac{1}{2} \\ \\ 1 & \text{for } x > \frac{1}{2} \end{cases}$$

20. The cumulative distribution function for a continuous random variable X is given below. Find its probability density function $f(x)$ for $0 \le x \le 1$.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \\ x^5 & \text{for } 0 \le x \le 1 \\ \\ 1 & \text{for } x > 1 \end{cases}$$

21. The number of years that a certain model of car will remain on the road (i.e. before it is scrapped), given that it has been on the road for 5 years, is a continuous random variable X with cumulative distribution given by

$$F(x) = \begin{cases} 0 & \text{for } x \le 5 \\ \\ 1 - \dfrac{25}{x^2} & \text{for } x > 5 \end{cases}$$

What is the probability that such a car will last longer than 10 years?