MATH1550, Winter 2023:
Exercise Set 4

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1. Suppose a coin is weighted so that the probability of getting heads on any flip is twice the probability of getting tails. The coin is tossed 3 times. Let X be the random variable which assigns total number of heads to an outcome.

   (a) Give the range of X and find $P(X = x)$ for each x in the range of X.

   (b) Find the cumulative distribution for X.

   (c) Draw a probability histogram for X.

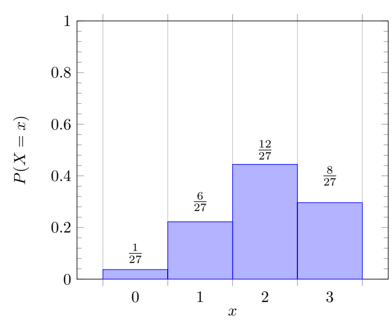
   Solution. (a) The range of X is the set $\{0, 1, 2, 3\}$.

   $$P(X = 0) = P(TTT) = \frac{1}{27},$$
   $$P(X = 1) = P(\{HTT, THT, TTH\}) = P(HTT) + P(THT) + P(TTH) = \frac{6}{27}$$
   $$P(X = 2) = P(\{HHT, HTH, THH\}) = P(HHT) + P(HTH) + P(THH) = \frac{12}{27}$$
   $$P(X = 3) = P(HHH) = \frac{8}{27}$$

   (b)

   $$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{27} & \text{for } 0 \le x < 1 \\ \frac{7}{27} & \text{for } 1 \le x < 2 \\ \frac{19}{27} & \text{for } 2 \le x < 3 \\ 1 & \text{for } x \ge 3 \end{cases}$$

   (c)



□

2. Which of the following functions can be used as a valid probability distribution function?

$$A: \quad f(x) = \frac{x-2}{5} \quad \text{for } x = 1, 2, 3, 4, 5$$

$$B: \quad f(x) = \frac{x^2}{30} \quad \text{for } x = 1, 2, 3, 4$$

$$C: \quad f(x) = \frac{x^2}{30} \quad \text{for } x = 0, 1, 2, 3, 4$$

$$D: \quad f(x) = \frac{1}{5} \quad \text{for } x = 0, 1, 2, 3, 4, 5$$

$$E: \quad f(x) = \frac{x}{15} \quad \text{for } x = 1, 2, 3, 4, 5$$

$$F: \quad f(x) = \frac{\binom{5}{x}}{32} \quad \text{for } x = 0, 1, 2, 3, 4, 5$$

Solution. The functions in B, C, E and F are valid; to see the check that $f(x) \geq 0$ for all x and $\sum_x f(x) = 1$. □

3. Determine an appropriate value for k so that

$$f(x) = \frac{k}{x} \quad \text{for } x = 1, 2, 3, 4, 5$$

is a valid probability distribution. *(Assume $f(x) = 0$ for all other values of x.)*

Solution. First note that whatever k we find must be positive. We require that

$$1 = \sum_{x=1}^{5} \frac{k}{x} = k \sum_{x=1}^{5} \frac{1}{x} = k \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = k \left(\frac{137}{60}\right)$$

which implies

$$k = \frac{60}{137}$$

□

4. A fair 4-sided die (with sides numbered $1, 2, 3, 4$) and a fair 8-sided die (with sides numbered $1, 2, 3, 4, 5, 6, 7, 8$) are rolled. Outcomes of the individual dice are independent. Let Y be the random variable that gives the sum of the two dice. Give the range and probability distribution of Y.

Solution. The sample space of this experiment is $\{(d_1, d_2) \mid d_1 \in \{1, 2, 3, 4\}, d_2 = \{1, 2, 3, 4, 5, 6, 7, 8\}\}$ with $4 \cdot 8 = 32$ equally likely outcomes.

The range of Y is the set $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

The probability distribution for Y is summarized below:

y	Outcomes	$P(Y=y)$
2	$(1,1)$	$\frac{1}{32}$
3	$(1,2),(2,1)$	$\frac{2}{32}$
4	$(1,3),(2,2),(3,1)$	$\frac{3}{32}$
5	$(1,4),(2,3),(3,2),(4,1)$	$\frac{4}{32}$
6	$(1,5),(2,4),(3,3),(4,2)$	$\frac{4}{32}$
7	$(1,6),(2,5),(3,4),(4,3)$	$\frac{4}{32}$
8	$(1,7),(2,6),(3,5),(4,4)$	$\frac{4}{32}$
9	$(1,8),(2,7),(3,6),(4,5)$	$\frac{4}{32}$
10	$(2,8),(3,7),(4,6)$	$\frac{3}{32}$
11	$(3,8),(4,7)$	$\frac{2}{32}$
12	$(4,8)$	$\frac{1}{32}$

□

5. Three (regular) dice are thrown and the $6^3 = 216$ possible outcomes are equally likely. Let X be the random variable whose value is the sum of the three dice. What is the range of X?

Solution. The range of X (all possible values for the sum of three dice) is:

$$\{3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18\}.$$

□

6. The cumulative distribution for discrete random variable X is

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{1}{3} & \text{for } x \in [1,4) \\ \frac{1}{2} & \text{for } x \in [4,6) \\ \frac{5}{6} & \text{for } x \in [6,10) \\ 1 & \text{for } x \geq 10 \end{cases}$$

(a) Find $P(X=4)$.

(b) Find $P(2 < X \leq 6)$.

Solution. (a)

$$P(X=4) = F(4) - F(1) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

(b)

$$P(2 < X \leq 6) = P(X \leq 6) - P(X \leq 2) = F(6) - F(2) = \frac{5}{6} - \frac{1}{3} = \frac{1}{2}.$$

□

7. Suppose the cumulative distribution for a random variable X is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \le x < 1 \\ \frac{5}{8} & 1 \le x < 2 \\ \frac{11}{12} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

(a) Give the probability distribution for X.
(b) Use $F(x)$ to find $P(\frac{1}{2} < X < \frac{5}{2})$.
(c) Draw a probability histogram for X.

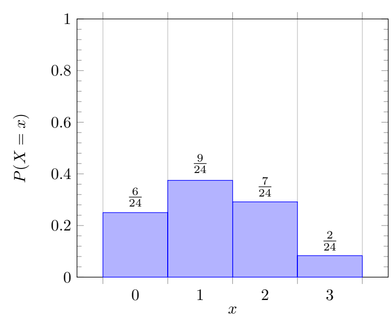
Solution. (a)

x	$P(X = x)$
0	$\frac{1}{4}$
1	$\frac{3}{8}$
2	$\frac{7}{24}$
3	$\frac{1}{12}$

(b) Using the cumulative distribution $F(x)$ for X we have

$$P\left(\frac{1}{2} < X < \frac{5}{2}\right) = F\left(\frac{5}{2}\right) - F\left(\frac{1}{2}\right) = \frac{11}{12} - \frac{1}{4} = \frac{2}{3}.$$

(c)



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8. A fair 4-sided die (with sides numbered $1, 2, 3, 4$) and a fair 6-sided die (with sides numbered $1, 2, 3, 4, 5, 6$) are rolled. Outcomes of the individual dice are independent. Let Y be the random variable that gives the sum of the two dice.

(a) What is range of Y?
(b) Give the probability distribution for Y (you don't need a formula).
(c) Give the cumulative distribution function for Y.

Solution. (a) The range of Y is $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
(b)

$$P(Y = 2) = \frac{1}{24}, \quad P(Y = 3) = \frac{2}{24}, \quad P(Y = 4) = \frac{3}{24}, \quad P(Y = 5) = \frac{4}{24}, \quad P(Y = 6) = \frac{4}{24},$$

$$P(Y = 7) = \frac{4}{24}, \quad P(Y = 8) = \frac{3}{24}, \quad P(Y = 9) = \frac{2}{24}, \quad P(Y = 10) = \frac{1}{24}$$

(c)

$$F(y) = \begin{cases} 0 & \text{for } y < 2 \\ \frac{1}{24} & \text{for } 2 \le y < 3 \\ \frac{3}{24} & \text{for } 3 \le y < 4 \\ \frac{6}{24} & \text{for } 4 \le y < 5 \\ \frac{10}{24} & \text{for } 5 \le y < 6 \\ \frac{14}{24} & \text{for } 6 \le y < 7 \\ \frac{18}{24} & \text{for } 7 \le y < 8 \\ \frac{21}{24} & \text{for } 8 \le y < 9 \\ \frac{23}{24} & \text{for } 9 \le y < 10 \\ 1 & \text{for } y \ge 10 \end{cases}$$

□

9. Two balls are chosen randomly without replacement from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings.

   (a) What is the range of X?
   (b) Find the probability distribution of X.
   (c) Find the cumulative distribution of X.

Solution. (a) The Range of X is $\{4, 2, 1, 0, -1, -2\}$

(b) The probability distribution for X is

$$P(X = 4) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$$

$$P(X = 2) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$$

$$P(X = 1) = \frac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \frac{32}{91}$$

$$P(X = 0) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$

$$P(X = -1) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$

$$P(X = -2) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$$

(c) The cumulative distribution of X is

$$F(x) = \begin{cases} 0 & \text{for } x < -2 \\ \frac{28}{91} & \text{for } -2 \le x < -1 \\ \frac{44}{91} & \text{for } -1 \le x < 0 \\ \frac{45}{91} & \text{for } 0 \le x < 1 \\ \frac{77}{91} & \text{for } 1 \le x < 2 \\ \frac{85}{91} & \text{for } 2 \le x < 4 \\ 1 & \text{for } x \ge 4 \end{cases}$$

□

10. Suppose discrete random variable X has range $\{0, 1, 2\}$ with probability distribution

$$f(x) = \frac{\binom{2}{x}\binom{4}{3-x}}{\binom{6}{3}}.$$

(a) Verify that this is a valid probability distribution.

(b) Create a histogram for this probability distribution.

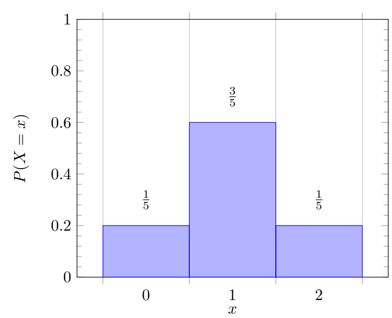
(c) Give the cumulative probability distribution for X.

(d) Come up with an example of a probability experiment which corresponds to this X.

Solution. (a) We can see that $f(x) \geq 0$ for all $x \in \{0, 1, 2\}$ since the expression for $f(x)$ involves only binomial coefficients, which are always positive. Next we note that

$$\sum_x f(x) = f(0) + f(1) + f(2) = \frac{\binom{2}{0}\binom{4}{3}}{\binom{6}{3}} + \frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} + \frac{\binom{2}{2}\binom{4}{1}}{\binom{6}{3}} = \frac{4}{20} + \frac{12}{20} + \frac{4}{20} = 1$$

(b)



(c)

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{5} & \text{for } 0 \leq x < 1 \\ \frac{4}{5} & \text{for } 1 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

(d) Consider the experiment of drawing 3 balls without replacement from a bag containing 2 gold balls and 4 silver balls. Let X be the random variable whose value is the number of gold balls drawn.

□

11. Suppose you have 5 cards which are numbered 1 to 5. You draw 2 of them at random without replacement. Let random variable X be the smallest number out the two cards you have drawn. Find $P(X = 2)$.

Solution. There are $\binom{5}{2} = 10$ different 2-card hands that can be made (not counting order). Of those 10 hands, there are 3 for which the smallest number is 2, namely $(2, 3), (2, 4)$ and $(2, 5)$, so

$$P(X = 2) = \frac{3}{10}.$$

□

12. In a certain dice rolling game, the player rolls two fair six-sided dice and wins \$3 if the sum of the dice is a multiple of 3, \$5 if the sum of dice is a multiple of 5 and \$7 if the sum of the dice is a multiple of 7. Let random variable Y denote the amount of money won on a single roll of both dice. Then Y has range $\{0, 3, 5, 7\}$. Find the probability distribution for Y.

    Fill in the blanks:

    $$P(Y = 0) = \_\_ \quad P(Y = 3) = \_\_ \quad P(Y = 5) = \_\_ \quad P(Y = 7) = \_\_$$

    Solution. Recall that the probability distribution for X, the sum of the dice is given by $f(x) = \frac{6-|7-x|}{36}$ for $x = 2, 3, \ldots, 12$. Then $Y = 3$ when $X = 3, 6, 9$, or 12, $Y = 5$ when $X = 5$, or 10, $Y = 7$ for $X = 7$, and $Y = 0$ for all other values of X. Thus

    $$P(Y = 3) = P(X = 3) + P(X = 6) + P(X = 9) + P(X = 12) = \frac{2}{36} + \frac{5}{36} + \frac{4}{36} + \frac{1}{36} = \frac{12}{36},$$

    $$P(Y = 5) = P(X = 5) + P(X = 10) = \frac{4}{36} + \frac{3}{36} = \frac{7}{36},$$

    $$P(Y = 7) = P(X = 7) = \frac{6}{36},$$

    $$P(Y = 0) = 1 - P(Y = 3) - P(Y = 5) - P(Y = 7) = 1 - \frac{25}{36} = \frac{11}{36}.$$

    □