MATH1550, Winter 2023:
Exercise Set 3

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1. Let A, B and C be events from a common sample space such that:

   $$P(A) = 0.72, \quad P(B) = 0.56, \quad P(C) = 0.25, \quad P(A \cap B) = 0.42, \quad P(A \cap C) = 0.18, \quad P(B \cap C) = 0$$

   Compute the following probabilities.

   (a) $P(A \cup B)$
   (b) $P(B \cup C)$
   (c) $P(B|A)$
   (d) $P(A \cup B \cup C)$

2. A pair of regular, six-sided dice are tossed. The sample space for this experiment is the following set of 36 ordered pairs
   $$S = \{(d_1, d_2)|d_1, d_2 \in \{1, 2, 3, 3, 5, 6\}\}$$
   where each outcome is equally likely.

   (a) List the elements for the following events:
   - $A = \{$The sum of the dice is 6$\}$
   - $B = \{$The first die shows 2$\}$
   - $C = \{$2 appears on at least one die$\}$
   - $D = \{$6 appears on at least one die$\}$

   (b) List the elements in the following events:
   - $A \cap B$
   - $A \cap C$
   - $A \cap D$
   - $B \cap C$
   - $B \cap D$
   - $C \cap D$

   (c) Compute the following conditional probabilities:
   - The probability that the sum of the dice is 6, given that the first die shows 2; i.e. $P(A|B)$.
   - The probability that the sum of the dice is 6, given that 2 appears on at least one die.
   - Find $P(C|A)$ and describe this event in words.
   - Find $P(A|D)$ and describe this event in words.
   - Find $P(B|C)$ and describe this event in words.
   - The probability that the first die shows 2 given that 6 appears on at least one die.
   - Find $P(C|D)$ and describe this event in words.

3. A coin is tossed 2 times. What is the probability that both flips are heads given that at least one of the flips is heads?

4. A coin is tossed 3 times. What is the probability that all three flips are heads given that the first flip is heads?

5. A coin is tossed 4 times. What is the probability that at least two consecutive flips are heads given that at least one flip is tails?

6. A coin is tossed 4 times. What is the probability that at least two consecutive flips are heads given that the third flip is heads?

7. Suppose you buy a lotto 649 ticket, and watch the drawing on t.v. What is the probability that you win the jackpot given that the first 3 numbers drawn match yours?

8. Two regular *fair* dice are rolled (i.e. 6-sided dice, where fair means that each side has an equally likely chance of coming up). What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

9. In a certain community, 36 percent of the families own a dog and 22 percent of the families that own a dog also own a cat. It is also known that 30 percent of all families own a cat.
   (a) What is the probability that a randomly selected family owns both a dog and a cat?
   (b) What is the conditional probability that a randomly selected family owns a dog given that it owns a cat?

10. At a certain dance studio, 90% of the students take ballet, 70% of the students take jazz, and 40% of the students take tap. It is known that 75% of those who take ballet are also in jazz, while 55% of those in jazz are also in tap. If a student is selected at random, what is the probability that they take both tap and jazz?

11. A total of 500 married working couples were polled about their annual salaries, with the following information resulting:

		husband less than $125,000	husband more than $125,000
wife	less than $125,000	212	198
wife	more than $125,000	36	54

    For instance, in 36 of the couples the wife earned more and then husband earned less than $ 125,000. If a couple is chosen at random:
    (a) What is the probability that the husband earns less than $125,000?
    (b) What is the conditional probability that the wife earns more than $125,000, given that the husband earns more than this amount?
    (c) What is the conditional probability that the wife earns more than $125,000, given that the husband earns less than this amount?

12. All job applicants for a certain teaching position are organized into the following table.

	Master's degree	No master's degree
3 or more years experience	18	9
Less than 3 years experience	36	27

    Let M be the event that a randomly selected applicant has a master's degree, and E be the event that that applicant has at least 3 years of experience. Find $P(E|M)$.

13. If A and B are events from sample space S, with $P(B) \neq 0$, then the definition of conditional probability says that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Suppose S is a finite set and that all outcomes of S are equally likely. Show that

$$P(A|B) = \frac{|A \cap B|}{|B|},$$

where $|X|$ denotes the number of elements in the sets X.

14. A penny, nickle and dime are tossed. Find the probability that they all show heads given that:

    (a) The penny is heads.

    (b) At least one of the coins is heads.

    (c) The dime is tails.

    Assume all outcomes are equally likely.

15. A billiard ball is drawn at random from a bag of 15 balls numbered 1 to 15.

    (a) What is the probability that ball drawn has number greater than 10?

    (b) If we know that the ball drawn has an even number, what is the probability its number is greater than 10? Express this as a conditional probability $P(A|B)$ with suitable events A and B.

16. A bin contains 25 red balls, 40 white balls and 35 black balls. What is the probability that one ball will be red and one ball will be white if the first ball is returned to the bin before drawing the second ball?

17. In a certain science degree program, 25 percent of the students failed their mathematics exam, 15 percent of the students failed their physics exam and 10 percent of the student failed both exams. A student is selected at random.

    (a) If the student failed physics, what is the probability that they failed mathematics?

    (b) If the student failed mathematics, what is the probability that they failed physics?

    (c) What is the probability that the student failed the mathematics or physics exam?

    (d) What is the probability that the student failed neither of these exams?

18. Let A and B be events in a common space with $P(A) = 0.6$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$. Find the following probabilities:

    (a) $P(A|B)$

    (b) $P(B|A)$

    (c) $P(A')$

    (d) $P(B')$

    (e) $P(A'|B')$

    (f) $P(B'|A')$

19. Two different digits are chosen at random from the digits 1 through 5.

    (a) If the sum of the two digits is odd, what is the probability that 2 is one of the digits that was chosen?

    (b) If 2 is one of the digits that was chosen, what is the probability that the sum of the two digits is odd?

20. A sample of people are surveyed. It is found that 65 percent of those people buy Sparkle-Sparkle laundry detergent, 40 percent buy Sparkle-Sparkle dish detergent and 20 percent buy both products. If a person at random is chosen from this sample, find the probability that:

    (a) The person buys Sparkle-Sparkle laundry detergent or Sparkle-Sparkle dish detergent.

    (b) The person buys Sparkle-Sparkle laundry detergent if they also buy Sparkle-Sparkle dish detergent.

    (c) The person buys neither product.

21. A blue die, a red die and a yellow die are rolled (assume these are regular 6-sided dice). Let B, R, Y denote the numbers appearing on the blue, red and yellow dice respectively.

    (a) What is the probability that no two dice land on the same number?

    (b) Given that no two dice land on the same number, what is the conditional probability that $B < R < Y$?

    (c) What is $P(B < R < Y)$?

22. An urn contains 4 white balls and 6 black balls. Suppose 3 balls are randomly drawn without replacement. What is the probability that the last ball drawn is white given the first two balls drawn are of the same colour?

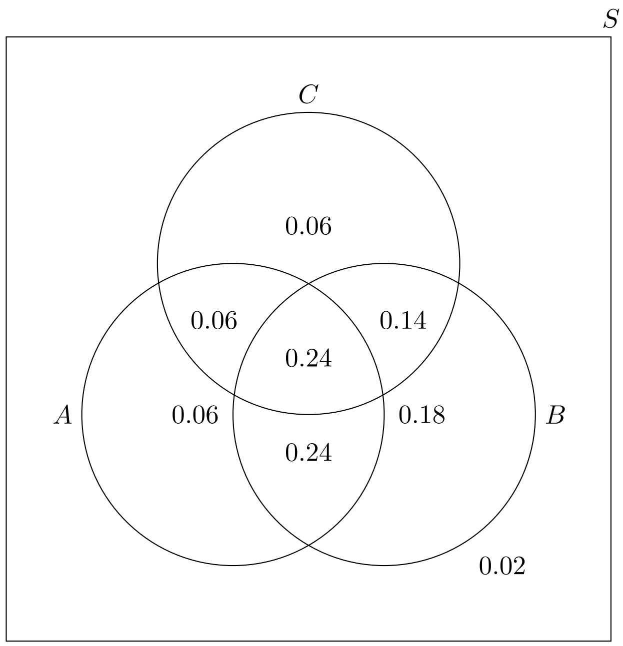
23. An urn contains 6 white balls and 9 black balls. Suppose 4 balls are randomly drawn without replacement. What is the probability that the first two drawn are white, and the last two drawn are black?

24. A roulette wheel has 38 spaces (all spaces are the same size); numbers 1-36, 0 and 00. Alice always bets that the ball will land on a number between 1-12. What is the probability that Alice will:

    (a) lose 5 times in a row.

    (b) lose twice and then win.

25. The following Venn diagram, shows events A, B and C (the three circles) in the sample space S. The values lying in each region are the probabilities of the subsets of S depicted by those regions.



Are events A, B and C independent?

26. A coin is tossed twice, and heads or tails is observed on each toss. Let

$$S = \{HH, HT, TH, TT\}$$

be the sample space for this experiment and consider the events

$$\begin{aligned} H_1 &= \{\text{heads appears on the first toss}\}, \\ H_2 &= \{\text{heads appears on the second toss}\}, \\ T_1 &= \{\text{tails appears on the first toss}\}, \\ T_2 &= \{\text{tails appears on the second toss}\}. \end{aligned}$$

(a) Suppose each outcome in S is equally likely. Show that successive coin tosses are independent events; i.e. show that $P(A \cap B) = P(A)P(B)$ where A and B are any successive pair of events from H_1, H_2, T_1, or T_2.

(b) Suppose the probability of each outcome in S is as follows:

$$P(\{HH\}) = 0.1, \quad P(\{HT\}) = 0.2, \quad P(\{TH\}) = 0.4, \quad P(\{TT\}) = 0.3.$$

Show that successive coin tosses are not independent in this case; this means $P(A \cap B) \neq P(A)P(B)$ for some pair of successive events A and B.

(c) Suppose the probability of each outcome in S is as follows:

$$P(\{HH\}) = 0.16, \quad P(\{HT\}) = 0.24, \quad P(\{TH\}) = 0.24, \quad P(\{TT\}) = 0.36.$$

Show that successive coin tosses are independent.

(d) Suppose the probability of each outcome in S is as follows:

$$P(\{HH\}) = 0.1, \quad P(\{HT\}) = 0.25, \quad P(\{TH\}) = 0.25, \quad P(\{TT\}) = 0.4.$$

Determine whether coin tosses independent in this case?

(e) Suppose that the coin is constructed so that the probability of getting heads on any toss is 0.3 and the probability of getting tails on any toss is 0.7. This is the same as assuming

$$P(H_1) = 0.3, \quad P(H_2) = 0.3, \quad P(T_1) = 0.7, \quad P(T_2) = 0.7.$$

If we assume that successive coin tosses are independent event, find
- $P(\{HH\})$
- $P(\{HT\})$
- $P(\{TH\})$
- $P(\{TT\})$

(f) Suppose that $P(H_1) \neq P(H_2)$ for some valid probability P on S. Is it possible that successive coin tosses are independent events?

27. Three machines M_1, M_2 and M_3 produce respectively $25, 35$ and 40 percent of the items in a factory. It is known that

- 3 percent of the items M_1 makes are defective,
- 2 percent of the items M_2 makes are defective, and
- 5 percent of the items M_3 makes are defective.

(a) What is the probability that a randomly selected item came from M_2?
(b) Find the probability that a randomly selected item is defective.
(c) Find the probability that a randomly selected item came from M_2 given that it is defective.

28. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the sample space for an experiment with equally likely outcomes and define events

$$A = \{1, 2, 3, 4\}, \quad B = \{2, 3, 4, 5\}, \quad C = \{4, 6, 7, 8\}.$$

Is $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$? Are A, B, C independent?

29. A box contains 3 different types of disposable flashlights. Suppose that 20% of the box is type 1, 30% is type 2 and 50% is type 3. The probability that type 1, type 2 and type 3 will give over 100 hours of use is respectively $0.70, 0.40$ and 0.30. What is the probability that a randomly chosen flashlight will give over 100 hours of use? If the selected flashlight lasted over 100 hours, what is the probability it was of type 2?

30. A total of 48 percent of the women, and 37 percent of the men who took a certain "quit smoking" class remained nonsmokers for at least one year after completing the class. These people then attended a success party at the end of a year. Assuming that everyone in the class was either a male or female, if 62 percent of the original class was male:

(a) What percentage of those attending the party were women?
(b) What percentage of the original class attended that party?

31. Two countries C_1 and C_2 produce respectively 40 and 60 percent of humanoid robots worldwide.
It is known that

- 2 percent of the humanoid robots that C_1 makes are not compatible with safety regulations, and
- 5 percent of the humanoid robots that C_2 makes are not compatible with safety regulations.

(a) Find the probability that a randomly selected humanoid robot was made in C_1.
(b) Find the probability that a randomly selected humanoid robot does not meet safety regulations.
(c) Find the probability that a randomly selected humanoid robot was made in C_1 given that it does not meet safety regulations.

32. A fair die (i.e. 6-sided die, where fair means that each side has an equally likely chance of coming up) is rolled 1000 times. What is the conditional probability that the last roll is a 6 given that none of the first 999 rolls result is a 6?

33. Let A and B be events with nonzero probability from a common sample space. If A and B are mutually exclusive, are they independent events?

34. Two probability experiments can be combined to form one experiment with a larger sample space; for example, the experiment of tossing a coin combined with rolling a die. If S_1 and S_2 are the sample spaces for the two probability experiments, we can form the new sample space with the Cartesian product

$$S_1 \times S_2 = \{(x, y) | x \in S_1, y \in S_2\}.$$

If A is an event in sample space S_1 (from the first experiment) we can identify the occurrence of A in the new experiment with the event $A \times S_2$ in $S_1 \times S_2$. Similarly, identify the event B from S_2 with $S_1 \times B$ in $S_1 \times S_2$. The event $A \times B = (A \times S_2) \cap (S_1 \times B)$ is when both A and B have occurred.

(a) A common way to assign probabilities to $S_1 \times S_2$ is to assume that the two experiments involved are independent of each other, and define

$$P(A \times B) = P(A)P(B),$$

where $P(A)$ and $P(B)$ are the probabilities from their individual experiments respectively. Then extend P to all of $S_1 \times S_2$ by assuming countable additivity. *(This way of assigning probabilities may be appropriate for the coin-toss die-roll experiment, where it is reasonable to assume that the two experiments don't interact.)*

Show that $(A \times S_2)$ and $(S_1 \times B)$ are independent events in the sample space $S_1 \times S_2$.

(b) Suppose S_1 and S_2 are finite, and all outcomes in S_1 and S_2 are equally likely (as individual experiments). If we specify that all outcomes of $S_1 \times S_2$ are equally likely, show that

$$P(A \times B) = P(A)P(B).$$

for any events $A \subseteq S_1$ and $B \subseteq S_2$.

(c) Suppose a fair coin is tossed, and then a pair of regular 6-sided dice are rolled. Assigning probabilities as in part (a), what is the probability that the coin shows tails and a sum of 8 is rolled on the dice.

(d) Bag 1 contains 5 red balls and 7 yellow balls. Bag 2 contains 12 balls numbered 1 through 12. Each ball has an equally likely chance of being drawn from its bag. Assuming that it is equally likely to choose any pair of balls, what is the probability of drawing a red ball along with a ball whose number is less than 5.

35. Suppose a coin is weighted so that $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$. The coin is tossed 3 times. Assuming that coin flips are independent events, what is the probability of each outcome of the 3-flip experiment?

36. In a certain geographical region during the month of April, it is known that the probability that a rainy day is followed by another rainy day is 0.8 and the probability that a sunny day is followed by a rainy day 0.6. If April 1st is a rainy day, find the probability that it will be rainy on April 3rd. *Assume it's sunny if it's not rainy.*

Solution. Let R_i be the event that it rains on the ith day of April, and S_i be the event that it is sunny on the ith day of April. The sample space can be partitioned into mutually exclusive events as $S = R_2 \cup R_2' = R_2 \cup S_2$. By the rule of total probability we have

$$P(R_3) = P(R_2) \cdot P(R_3|R_2) + P(S_2) \cdot P(R_3|S_2) = (0.8)(0.8) + (0.2)(0.6) = 0.76.$$

$\square$

37. Persons A, B and C are responsible for shipping orders from a warehouse for a certain online retailer. Person A fills 30% of all orders, person B fills 40% of all orders and person C fills the remaining 30% of the orders. Based on past experience it is known that person A makes a mistake in an order 1% of the time, person B makes a mistake 5% of the time and person C makes a mistake 3% of the time.

    An email complaint is received about a mistake in an order. What is the probability that mistake was made by person C?