MATH1550, Winter 2023:
Exercise Set 2

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1. Rolling a pair of dice with equally likely outcomes, what is the probability of rolling a 5, a 7 or a 12? (*Here we want the sum of the two dice.*)

   Solution. There are 36 possible outcomes of two dice; 6 outcomes for one die and 6 outcomes for the other, which makes $6 \cdot 6$ possible pairings. The sample space has the following outcomes:

$$\begin{gathered}(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6).\end{gathered}$$

   Let A_5, A_7 and A_{12} be the events of rolling a 5, 7, and 12 respectively. Then

$$\begin{gathered}A_5 = \{(1,4),(2,3),(3,2),(4,1)\}\\A_7 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}\\A_{12} = \{(6,6)\}\end{gathered}$$

   Since each outcome is equally likely, the probability assigned to any event is the ratio of successful outcomes to total outcomes. Thus

$$P(A_5 \cup A_7 \cup A_{12}) = P(\{(1,4),(2,3),(3,2),(4,1),(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(6,6)\}) = \frac{11}{36}.$$

   $\square$

2. Rolling a pair of dice with equally likely outcomes, what is the probability of rolling a 1, a 4 or an 11? (*Here we want the sum of the two dice.*)

   Solution. Let A_1, A_4 and A_{11} be the events of rolling a 1, 4, and 11 respectively. Then

$$\begin{gathered}A_1 = \emptyset\\A_4 = \{(1,3),(2,2),(3,1)\}\\A_{11} = \{(5,6),(6,5)\}\end{gathered}$$

   Since each outcome is equally likely, the probability assigned to any event is the ratio of successful outcomes to total outcomes. Thus

$$P(A_1 \cup A_4 \cup A_{11}) = P(\{(1,3),(2,2),(3,1),(5,6),(6,5)\}) = \frac{5}{36}.$$

   $\square$

3. Rolling a pair of dice with equally likely outcomes, what is the probability of rolling at least 8? (*Here we want the sum of the two dice.*)

   Solution. The following 15 outcomes have a sum of at least 8: $(2,6)$, $(3,5)$, $(3,6)$, $(4,4)$, $(4,5)$, $(4,6)$, $(5,3)$, $(5,4)$, $(5,5)$, $(5,6)$, $(6,2)$, $(6,3)$, $(6,4)$, $(6,5)$, $(6,6)$. Therefore

$$P(\text{sum at least 8}) = \frac{15}{36} = \frac{5}{12}.$$

   $\square$

4. An experiment has 5 possible outcomes S_1, S_2, S_3, S_4 and S_5 (which are mutually exclusive). Which of the following are valid ways to assign probabilities?

$$A: \quad P(S_1) = 0.20, \quad P(S_2) = 0.20, \quad P(S_3) = 0.20, \quad P(S_4) = 0.20, \quad P(S_5) = 0.20$$

$$B: \quad P(S_1) = 0.21, \quad P(S_2) = 0.26, \quad P(S_3) = 0.58, \quad P(S_4) = 0.01, \quad P(S_5) = 0.06$$

$$C: \quad P(S_1) = 0.18, \quad P(S_2) = 0.19, \quad P(S_3) = 0.20, \quad P(S_4) = 0.21, \quad P(S_5) = 0.22$$

$$D: \quad P(S_1) = 0.10, \quad P(S_2) = 0.30, \quad P(S_3) = 0.10, \quad P(S_4) = 0.60, \quad P(S_5) = -0.10$$

$$E: \quad P(S_1) = 0.23, \quad P(S_2) = 0.12, \quad P(S_3) = 0.05, \quad P(S_4) = 0.5, \quad P(S_5) = 0.08$$

Solution. The functions P in A, C, only (check that the given probabilities are non-negative and sum to 1).

□

5. An experiment has four possible outcomes A, B, C, D that are mutually exclusive. Are either of the following assignments of probabilities permissible?

   (a) $P(A) = 0.72, P(B) = 0.13, P(C) = 0.25, P(D) = -0.10$
   (b) $P(A) = \frac{3}{111}, P(B) = \frac{65}{111}, P(C) = \frac{22}{111}, P(D) = \frac{21}{111}$

Solution. (a) Not permissible since there is a negative probability.

(b) This probability distribution is permissible, since all probabilities are positive, and the total probability adds to 1 (we must assume countable additivity).

□

6. For events A and B we have that $P(A) = 0.45$, $P(B) = 0.33$ and $P(A \cap B) = 0.12$. Find $P(A \cup B)$.

Solution. By the inclusion-exclusion principle:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.45 + 0.33 - 0.12 = 0.66.$$

□

7. Suppose a random experiment has sample space $S = \{a, b, c, d\}$ with

$$P(a) = 0.2, \quad P(b) = 0.3, \quad P(c) = 0.4, \quad P(d) = 0.1.$$

Consider the events $A = \{a, b\}$ and $B = \{b, c, d\}$ and determine the following probabilities:

   (a) $P(A)$
   (b) $P(B)$
   (c) $P(A')$
   (d) $P(A \cup B)$
   (e) $P(A \cap B)$

Solution. (a)

$$P(A) = P(a) + P(b) = 0.2 + 0.3 = 0.5$$

(b)

$$P(B) = P(b) + P(c) + P(d) = 0.3 + 0.4 + 0.1 = 0.8$$

(c)

$$P(A') = 1 - P(A) = 1 - 0.5 = 0.5$$

or

$$P(A') = P(\{c, d\}) = P(c) + P(d) = 0.4 + 0.1 = 0.5$$

(d)

$$P(A \cup B) = P(\{a, b, c, d\}) = P(S) = 1$$

(e)

$$P(A \cap B) = P(b) = 0.3$$

□

8. For events A and B we have that $P(A) = 0.72$, $P(B) = 0.19$ and $P(A \cap B) = 0.08$. Find $P(A' \cap B')$.

Solution. By DeMorgan's law $A' \cap B' = (A \cup B)'$, and so $P(A' \cap B') = P((A \cup B)')$. Then using the rule for probability of a union of two events we have:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.72 + 0.19 - 0.08 = 0.83.$$

The rule for probability of the complement says $P(A') = 1 - P(A)$. Therefore

$$\begin{aligned} P(A' \cap B') &= P((A \cup B)') \\ &= 1 - P(A \cup B) \\ &= 1 - 0.83 \\ &= 0.17. \end{aligned}$$

□

9. Given $P(A) = 0.59$, $P(B) = 0.3$ and $P(A \cap B) = 0.21$ find $P(A \cap B')$.

Solution. It may help to draw Venn diagrams here. Note that $A \cap B' = \{x | x \in A \text{ and } x \notin B\} = A \setminus B$. The sets $A \setminus B$ and $A \cap B$ are disjoint (i.e. events are mutually exclusive) and so $A = (A \setminus B) \cup (A \cap B)$ is a partition of A. Thus

$$P(A) = P(A \setminus B) + P(A \cap B) \quad \Rightarrow \quad P(A \setminus B) = P(A) - P(A \cap B) = 0.59 - 0.21 = 0.38.$$

So $P(A \cap B') = 0.38$.

□

10. A 52-card deck is divided into 13 *ranks*, $A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$, each having 4 *suits*, $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$. What is the probability of drawing a six-card hand from a standard deck of 52 cards with: four-of-a-kind, and one pair?

Solution. The probability of obtaining this type of hand is

$$\frac{13 \cdot \binom{4}{4} \cdot 12 \cdot \binom{4}{2}}{\binom{52}{6}} = \frac{936}{20358520} \approx 0.00046$$

Explanation: Choose the rank of the four-of-a-kind first (13 options). Then choose 4 of the 4 suits within that rank ($\binom{4}{4}$ options). Choose the rank of the pairs from the 12 remaining ranks. Choose 2 of the 4 suits ($\binom{4}{2}$ options) for the pair.

□

11. What is the probability of drawing a five-card poker hand of: two (different) pairs?

Solution. The probability of obtaining this type of hand is

$$\frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\cdot 44}{\binom{52}{5}} = \frac{123552}{2598960} \approx 0.0475$$

Explanation: Choose two ranks to be the pairs (for example aces and fives). Then choose 2 of the 4 suits for the first pair, and 2 of the 4 suits for the second pair. Finally choose 1 of the 44 remaining cards for the fifth card (this card cannot have the same rank as either pair, to avoid getting a three-of-a-kind).

□

12. What is the probability of drawing a seven-card hand from a deck of 52 cards with: three-of-a-kind, and two (different) pairs?

Solution. The probability of obtaining this type of hand is

$$\frac{13\cdot\binom{4}{3}\cdot\binom{12}{2}\cdot\binom{4}{2}\cdot\binom{4}{2}}{\binom{52}{7}} = \frac{123552}{133784560} = 0.0009$$

Explanation: We have a choice of 13 ranks for the three-of-a-kind. From those choose 3 of the 4 suits. Then choose 2 of the 12 remaining ranks to make the two pairs. For each pair choose 2 of the 4 suits. □

13. Two cards are randomly drawn from a standard deck of 52 cards. What is the probability the rank of both cards will be greater than 3 but less than 8?

Solution. We are allowed to draw, 4's, 5's, 6's or 7's, so there are $4\times 4 = 16$ successful cards, of which we choose 2. There are $\binom{52}{2}$ different 2-card hands in total. So the probability that the rank of both cards will be greater than 3 but less than 8 is

$$\frac{\binom{16}{2}}{\binom{52}{2}} = \frac{120}{1326} \approx 0.0905.$$

□

14. For events A, B and C we have that $P(A) = 0.62$, $P(B) = 0.31$, $P(C) = 0.50$, $P(A\cap B) = 0.09$, $P(A\cap C) = 0.34$, $P(B\cap C) = 0.16$ and $P(A\cap B\cap C) = 0.01$. Find $P(A\cup B\cup C)$.

Solution. By the inclusion-exclusion principle:

$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$$

$$= 0.62+0.31+0.50-0.09-0.34-0.16+0.01 = 0.85$$

□

15. Six regular dice are thrown at the same time. *(regular means six-sided with each outcome equally likely)*

(a) Describe the sample space of this experiment and count how many possible outcomes there are. *(you might wish to think of the dice as being six different colours)*

(b) What is the probability of rolling a *straight*, i.e. ⚀⚁⚂⚃⚄⚅? *(in any order)*

(c) What is the probability of rolling a *full house*; three-of-a-kind with two-of-(another)kind, plus a sixth number different form those two kinds? For example ⚀⚀⚀⚁⚁⚂. *(in any order)*

Solution. (a) The sample space can be described as

$$\{(d_1, d_2, d_3, d_4, d_5, d_6)|d_1, d_2, d_3, d_4, d_5, d_6 \in \{1, 2, 3, 4, 5, 6\}\}.$$

There are $6^6 = 46656$ elements in this set.

(b) A straight can be made in 6! ways (counts the different ways it can appear on the six coloured dice), so it has probability

$$\frac{6!}{6^6} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6} = \frac{5}{324} \approx 0.0154$$

(c) There are 6 kinds (⚀,⚁,⚂,⚃,⚄, or ⚅) to choose for the three-of-a-kind, leaving 5 kinds to choose for the two-of-kind and 4 kinds remaining for the sixth die. So $6 \cdot 5 \cdot 4 = 120$ counts the different types of full houses that can be made; however it does not include the multiple ways each type can be rolled with the six dice.

The number of ways that each of these full house types can be rolled is

$$\frac{6!}{3! \cdot 2!} = 60.$$

(recall permutations with repeated letters example AAABBC). Therefore the probability of rolling a full house is

$$\frac{6 \cdot 5 \cdot 4 \cdot 60}{6^6} = \frac{6 \cdot 5 \cdot 4 \cdot 6 \cdot 5 \cdot 2}{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6} = \frac{25}{162} \approx 0.1543$$

□

16. Three coins, with sides marked H and T, are tossed.

(a) List all possible outcomes. What is the probability of any outcome given that they are all equally likely?

(b) If we are only interested in the total number of heads that appear in this experiment, then we can take the sample space to be $S = \{0, 1, 2, 3\}$. Given that the outcomes of part (a) are equally likely, what are the probabilities for each element of S?

Solution. (a) The set of all possible outcomes can be expressed as

$$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Since there are 8 equally likely outcomes, each outcome has probability $\frac{1}{8}$ or 0.125.

(b) We can summarize these probabilities in a table

x	$P(x)$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

$\square$

17. Alice, Bob, Celia and Dan apply for job opening, where one of the four will be selected. Alice is twice as likely to be selected as Bob. Bob and Celia have an equally likely change of being selected, while Celia is twice as likely to be selected as Dan. What is the probability that Alice will not be selected?

Solution. If $P(\text{Dan}) = x$ then $P(\text{Bob}) = P(\text{Celia}) = 2x$ and $P(\text{Alice}) = 2P(\text{Bob}) = 4x$. Then

$$1 = P(\text{Alice, Bob, Celia, Dan}) = P(\text{Alice}) + P(\text{Bob}) + P(\text{Celia}) + P(\text{Dan}) = 4x + 2x + 2x + x = 9x$$

So $x = \frac{1}{9}$ and hence

$$P(\text{not Alice}) = 1 - P(\text{Alice}) = 1 - \frac{4}{9} = \frac{5}{9}.$$

$\square$

18. Three horses, A, B, and C, are in a race. Horse A is twice as likely to win as B, and B is twice as likely to win as C.

(a) Find the probabilities $P(A)$, $P(B)$, and $P(C)$ of each horse winning.

(b) What is the probability that B or C wins?

Solution. (a) By the postulates of probability we have $P(A)+P(B)+P(C) = 1$. Since $P(B) = 2P(C)$ and $P(A) = 2P(B) = 4 \cdot P(C)$ we have

$$4P(C) + 2P(C) + P(C) = 1 \quad \Rightarrow \quad 7P(C) = 1 \quad \Rightarrow \quad P(C) = \frac{1}{7}$$

and thus $P(A) = \frac{4}{7}$ and $P(B) = \frac{2}{7}$.

(b) The probability that B or C wins, or $P(B \cup C)$, is $P(B)+P(C)+\frac{3}{7}$. This is also $P(A') = 1-P(A)$.

$\square$

19. A regular coin and 6-sided die are tossed. We will express the sample space for this experiment with the set

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

(a) Express each of the following events as subsets of S:
- $A = \{\text{heads with an even number}\}$
- $B = \{\text{the number is less than } 3\}$
- $C = \{\text{tails and an odd number}\}$

(b) If all outcomes in S are equally likely, find $P(A)$, $P(B)$ and $P(C)$.

(c) Express the following events as subsets of S:
- A or B occurs
- B and C occur.
- Only B occurs.

(d) Which pairs of events A, B and C are mutually exclusive?

Solution. (a) Express each of the following events as subsets of S:
- $A = \{H2, H4, H6\}$
- $B = \{H1, H2, T1, T2\}$
- $C = \{T1, T3, T5\}$

(b) We have $P(A) = \frac{3}{12} = \frac{1}{4}$, $P(B) = \frac{4}{12} = \frac{1}{3}$, and $P(C) = \frac{3}{12} = \frac{1}{4}$.

(c) Express the following events as subsets of S:

  - The event that A or B occurs is $A \cup B = \{H1, H2, H4, H6, T1, T2\}$
  - The event that B and C occur is $B \cap C = \{T1\}$
  - The event that only B occurs is $B \cap (A \cup C)' = (B \setminus A) \setminus C = \{H1, T2\}$.

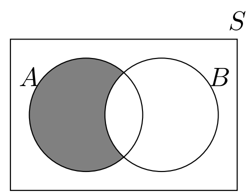
(d) Note that $A \cap B = \{H_2\}$, $A \cap C = \emptyset$ and $B \cap C = \{T1\}$, so we see that only A and C are mutually exclusive sets.

□

20. Let A and B be events from a sample space S. Find an expression (using unions, intersections and complements) for the following events, and draw a Venn diagram representing the event.

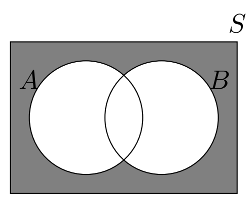
  - A but not B
  - Neither A nor B.
  - Either A or B but not both.

Solution. • The event "A but not B" is the same as $A \cap B'$.



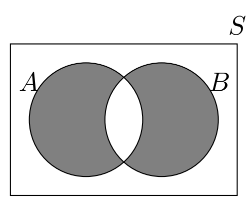
$A \cap B'$

  - The event "neither A nor B" is the same as $A' \cap B'$ or $(A \cup B)'$ by DeMorgan's Law.



$(A \cup B)'$

  - The event "either A or B but not both" is the same as $(A \cup B) \setminus (A \cap B)$, or $(A \cup B) \cap (A \cap B)'$ or $(A \cap B') \cup (B \cap A')$.



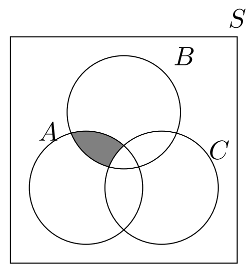
$A \cap B$

□

21. Let A, B and C be events from a sample space S. Find an expression (using unions, intersections and complements) for the following events, and draw a Venn diagram representing the event.

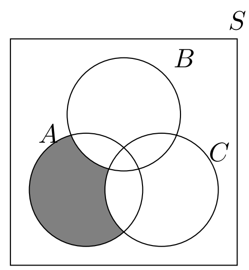
  - A and B but not C occur
  - Only A occurs

Solution. • The event "A and B but not C" is the same as $(A \cap B) \setminus C$ or $(A \cap B) \cap C'$.



$(A \cap B) \cap C'$

- The event "only A" is the same as $(A \setminus B) \setminus C$ or $(A \cap B') \cap C'$.



$(A \cap B) \cap C'$

□

22. A six-sided die, with sides numbered 1 through 6, is weighted so that the probability of rolling a given number is as follows:

x	$P(x)$
1	0.1
2	0.3
3	0.2
4	0.1
5	0.1
6	0.2

(a) Verify that P is a valid probability measure for the experiment (assuming countable additivity).

(b) Consider the following events:

$$A = \{\text{even number}\}, \quad B = \{2, 3, 4, 5\}, \quad C = \{x : x < 3\}, \quad D = \{x | x > 7\}.$$

Find:

- $P(A)$
- $P(B)$
- $P(C)$
- $P(D)$

Solution. (a) Let $S = \{1, 2, 3, 4, 5, 6\}$ be the sample space. If A is any event (subset) of S then $P(A)$ equals the sum of the probabilities of the elements of A (or $P(A) = \sum_{a \in A} P(a)$) by the postulate of countable additivity. Since $P(x) \geq 0$ for each $x \in S$, it follows that $P(A) \geq 0$ for any event A. Furthermore we have that

$$P(S) = P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 0.1 + 0.3 + 0.2 + 0.1 + 0.1 + 0.2 = 1.$$

Therefore P is indeed a probability measure on S.

(b)
- $P(A) = P(\{2,4,6\}) = P(2) + P(4) + P(6) = 0.3 + 0.1 + 0.2 = 0.6.$
- $P(B) = P(2) + P(3) + P(4) + P(5) = 0.3 + 0.2 + 0.1 + 0.1 = 0.7.$
- $P(C) = P(\{1,2\}) = P(1) + P(2) = 0.1 + 0.3 = 0.4.$
- $P(D) = P(\emptyset) = 0.$

□

23. Five valentines are randomly placed into five letter boxes. What is the probability that exactly two of the five boxes will be empty?

Solution. We calculate the probability as number of successful outcomes divided by total number of outcomes. Each of the 5 valentines has a choice of 5 boxes to be placed in, for a total of $5^5 = 3125$ outcomes.

To count "successful" outcomes, first choose 2 of 5 boxes to be empty. We then choose 1 of the 3 remaining boxes to hold 3 valentines while the other 2 boxes have 1 each, or choose 2 of the 3 remaining boxes to have 2 valentines each, while the other box has 1. In the first case choose 3 of 5 valentines for the 3-box, choose 1 of 2 for the next box, leaving 1 for the last box. In the second case choose 2 of 5 for the first 2-box, choose 2 of 3 for the second 2-box, leaving 1 for the other box. So the number of "successful" outcomes is,

$$\binom{5}{2}\cdot\binom{3}{1}\cdot\binom{5}{3}\cdot\binom{2}{1}\cdot\binom{1}{1}+\binom{5}{2}\cdot\binom{3}{2}\cdot\binom{5}{2}\cdot\binom{3}{2}\cdot\binom{1}{1} = 1500$$

Therefore the probability that exactly two of the five boxes will be empty is

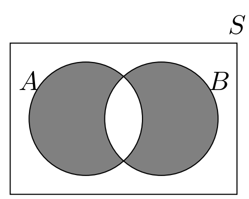
$$\frac{1500}{3125} = 0.48.$$

□

24. There is a 0.08 probability that Andy is absent from work, and a 0.05 probability that Blake is absent from form work. The probability that both will be absent is 0.02. What is the probability that only one of the two will be absent at work?

Solution. If A and B are the events that Andy and Blake are absent respectively, then the event that only one of the two will be absent is

$$(A \cup B) \setminus (A \cap B)$$



Since

$$(A \cup B) = ((A \cup B) \setminus (A \cap B)) \cup (A \cap B)$$

is a disjoint union, we have

$$P(A \cup B) = P((A \cup B) \setminus (A \cap B)) + P(A \cap B)$$

and so

$$P((A \cup B) \setminus (A \cap B)) = P(A \cup B) - P(A \cap B) = (P(A) + P(B) - P(A \cap B)) - P(A \cap B).$$

Therefore the probability that only one of the two will be absent at work is

$$P(A) + P(B) - 2P(A \cap B) = 0.08 + 0.05 - 2(0.02) = 0.09.$$

$\square$