MATH1550, Winter 2023:
Exercise Set 11

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1. If X is a Bernoulli random variable, where θ is the probability of a success, show that

   (a) $\mu = \theta$

   (b) $\sigma^2 = \theta(1-\theta)$.

   Solution. (a) $\mu = 1 \cdot \theta + 0 \cdot (1-\theta) = \theta$.

   (b) Using the definition of variance,

$$\begin{aligned} \sigma^2 &= (1-\mu)^2 \cdot \theta + (0-\mu)^2 \cdot (1-\theta) \\ &= (1-\theta)^2 \cdot \theta + (0-\theta)^2 \cdot (1-\theta) \\ &= \left(\theta \cdot (1-\theta) + (0-\theta)^2\right) \cdot (1-\theta) \\ &= \theta(1-\theta). \end{aligned}$$

□

2. If X is a Bernoulli random variable, where θ is the probability of a success. Find the cumulative distribution function $F(x)$ for X.

   Solution.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-\theta & \text{for } 0 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

□

3. Let X be a random variable with binomial distribution $b(x; n, \theta)$. Prove that

$$b(x; n, \theta) = b(n-x; n, 1-\theta).$$

   Solution. A simple non-technical explanation for this identity can be seen by observing that $b(x; n, \theta)$ (the left-side of the identity) is the probability of x successes in n trials with success probability θ. If in n trials there are x successes, then this means $n-x$ trials were unsuccessful, the probability of this is $b(n-x; n, 1-\theta)$ (the right-side of the identity). Therefore, the two sides are equal.

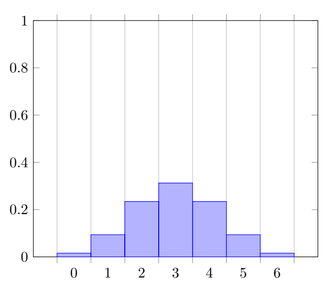
   Alternatively, we can use the formula: $b(x; n, \theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$. Then,

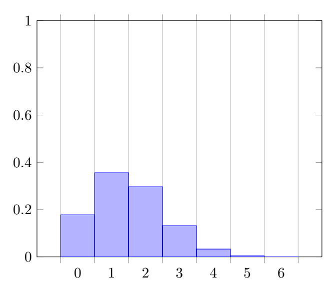
$$b(n-x; n, 1-\theta) = \binom{n}{n-x}(1-\theta)^{n-x}\theta^x.$$

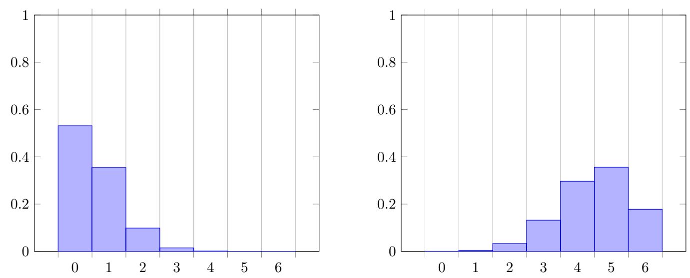
   Since $\binom{n}{x} = \binom{n}{n-x}$ (see Chapter 1), the expressions for $b(x; n, \theta)$ and $b(n-x; n, 1-\theta)$ are equal.

□

4. For each graph of $b(x; n, \theta)$ we have $n = 6$. Determine which of these has $\theta = 0.1, 0.25, 0.5$, and 0.75







Solution. A random variable with binomial distribution has $\mu = n\theta$.

When $\theta = 0.5$, we have $\mu = 6 \cdot 0.5 = 3$. We expect that the corresponding histogram is centered symmetrically around 3. Hence, the top-left histogram corresponds to $\theta = 0.5$.

When θ is larger than 0.5, we expect more successes than failures. Therefore, the histogram will be accumulated to the right of 3 (the mean when $\theta = 0.5$). Hence, the bottom-right histogram corresponds to $\theta = 0.75$.

Using similar arguments we can conclude that the bottom-left histogram corresponds to $\theta = 0.1$, and the top-right histogram corresponds to $\theta = 0.25$. $\square$

5. Let X have binomial distribution $b(x; 12, 0.25)$. Find

   (a) $P(X = 6)$

   (b) $P(3 \le X \le 5)$

Solution. (a) Recall that binomial distribution $b(x; n, \theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, for $x = 0, 1, \ldots, n$. So we have

$$P(X = 6) = b(6; 12, 0.25) = \binom{12}{6}(0.25)^6(0.75)^6 \approx 0.0401$$

(b)

$$\begin{aligned} P(3 \le X \le 5) &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= \binom{12}{3}(0.25)^3(0.75)^9 + \binom{12}{4}(0.25)^4(0.75)^8 + \binom{12}{5}(0.25)^5(0.75)^7 \\ &\approx 0.5549. \end{aligned}$$

$\square$

6. Let X have binomial distribution $b(x; 30, 0.3)$ find $P(X = 10)$.

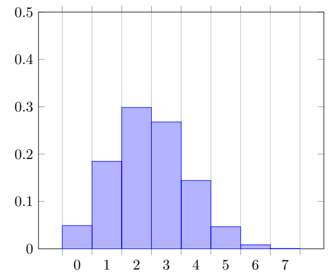
Solution.

$$P(X = 10) = b(10; 30, 0.3) = \binom{30}{10}(0.3)^{10}(0.7)^{20} \approx 0.1416.$$

$\square$

7. Draw the probability histogram for a random variable X with binomial distribution $b(x; 7, 0.35)$. (*You will need to use approximate values here.*)

Solution.



□

8. You are fishing on your favourite lake. For a single cast of your fishing rod you know there is a 45% chance of catching a walleye, a 25% chance of catching a pike, and a 30% chance of catching a bass. You are patient and always catch something on a cast.

   (a) What is the probability of catching exactly 12 bass on 20 casts?

   (b) How many walleye do you expect to catch in 20 casts?

   (c) What is the probability that you catch exactly 8 pike in 15 casts?

   Solution. (a) Let X be a random variable with binomial distribution $b(x; 20, 0.3)$ $(x = 0, 1, \ldots, 20)$. We are interested in the value of $P(X = 12)$.

   $$P(X = 12) = \binom{20}{12}(0.3)^{12}(1 - 0.3)^{20-12} = \binom{20}{12}(0.3)^{12}(0.7)^{8} \approx 0.00386.$$

   (b) Let X be a random variable with binomial distribution $b(x; 20, 0.45)$ $(x = 0, 1, \ldots, 20)$. We are interested in the value of μ. We know that for random variables with binomial distribution $\mu = n\theta$. So, $\mu = 20 \cdot 0.45 = 9$.

   (c) Let X be a random variable with binomial distribution $b(x; 15, 0.25)$ $(x = 0, 1, \ldots, 15)$. We are interested in the value of $P(X = 8)$.

   $$P(X = 8) = \binom{15}{8}(0.25)^{8}(1 - 0.25)^{15-8} = \binom{15}{8}(0.25)^{8}(0.75)^{7} \approx 0.01311.$$

   □

9. Suppose that it is known that on any given day in the month of March there is a 0.3 probability of rain. Find the mean number of rainy days in March and the standard deviation.

   Solution. We can think of rainy days in March as a random variable X with binomial distribution $b(x; 31, 0.3)$ $(x = 0, 1, \ldots, 31)$.

   Then, the mean is $\mu = n\theta = 31 \cdot 0.3 = 9.3$, and the variance is

   $$\sigma^2 = n\theta(1 - \theta) = 31 \cdot 0.3 \cdot (1 - 0.3) = 6.51.$$

   Finally, the standard deviation is $\sqrt{6.51} \approx 2.55$. □

10. Your friend claims that if she rolls a regular 6-sided die 6 times, it is more likely than not that at least one time the outcome will be a 6. Is she right?

Solution. We can think of this experiment as a random variable X with binomial distribution $b\left(x; 6, \frac{1}{6}\right)$ $(x = 0, 1, \ldots, 6)$ where x is the number of times your friend rolls a 6. The probability that at least one roll is a 6 is $P(X \geq 1) = 1 - P(X = 0)$. We have

$$P(X = 0) = \binom{6}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6 \approx 0.3349,$$

so

$$P(X \geq 1) = 1 - P(X = 0) \approx 1 - 0.3349 = 0.6651.$$

This means your friend is right in her claim. □

11. Suppose X has discrete uniform distribution, where the range of X is $\{1, 2, \ldots, 24\}$. Find the mean of X.

Solution. Since X has discrete uniform distribution we have that $f(x) = \frac{1}{24}$ for each $x \in \{1, 2, \ldots, 24\}$. The mean of X is

$$\begin{aligned}\mu &= \sum_x x f(x) \\ &= \frac{\sum_{x=1}^{24} x}{24} \\ &= \frac{300}{24}\end{aligned}$$

□

12. Suppose X has discrete uniform distribution, where the range of X is $\{0, 2, 4, 6, \ldots, 2k\}$ for some positive integer k. Find the mean of X.

Solution. Since the range of X has $k+1$ elements, we have that $f(x) = \frac{1}{k+1}$ for each $x \in \{0, 2, 4, 6, \ldots, 2k\}$. The mean of X is

$$\begin{aligned}\mu &= \sum_x x f(x) \\ &= \frac{\sum_{i=0}^{k} 2i}{k+1} \\ &= \frac{2\sum_{i=0}^{k} i}{k+1} \\ &= \frac{2(\frac{1}{2}k(k+1))}{k+1} \\ &= k\end{aligned}$$

□

13. A multiple choice test has 8 questions where there are 3 options, A, B or C, per question (and only 1 is correct). If a student decides to write the test by choosing A, B or C at random for each question, what is the probability that they get exactly 4 questions correct?

Solution. Let X be the number of questions that the student gets correct. Then X is a binomial random variable with $n = 8$ and $\theta = \frac{1}{3}$. The probability of getting exactly 4 correct answers is

$$b\left(4; 8, \frac{1}{3}\right) = \binom{8}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^4 = 70\left(\frac{1}{81}\right)\left(\frac{16}{81}\right) = \frac{1120}{6561}.$$

□

14. A dart player can hit the bullseye with a probability of 0.25 on a given shot. What is the probability that they will hit the bullseye more than 4 times in 6 shots.

Solution. Let X be the number of darts that hit the bullseye in 6 shots. Then X is a binomial random variable with $n = 6$ and $\theta = 0.25$. The probability that they will hit the bullseye more than 4 times is

$$b(5;6,0.25) + b(6;6,0.25) = \binom{6}{5}(0.25)^5(0.75) + \binom{6}{6}(0.25)^6) = \frac{18}{4096} + \frac{1}{4096} = \frac{19}{4096}.$$

□

15. A dart player can hit the bullseye with a probability of 0.25 on a given shot. What is the minimum number of darts that must be thrown, so that there is at least a 50% probability that the bullseye gets hit. *Hint: Consider the probability that the bullseye does not get hit.*

Solution. The probability that the player will miss the bullseye on a given shot is $1 - 0.25 = 0.75$. If the player takes n shots, the probability they will miss all n shots is

$$b(n;n,0.75) = \binom{n}{n}(0.75)^n(0.25)^0 = (0.75)^n.$$

Therefore the probability that at least one shot will hit the bullseye in n shots is

$$1 - (0.75)^n.$$

We want the smallest n such that

$$1 - (0.75)^n \geq 0.5 \quad \Rightarrow \quad 0.5 \geq (0.75)^n.$$

By trial, we have that $(0.75)^1 > 0.5$ and $(0.75)^2 = \frac{9}{16} > 0.5$ but $(0.75)^3 = \frac{27}{64} < 0.5$. So $n = 3$ is the minimum number of darts that must be thrown.

Solving with logarithms: We have $0.5 \geq (0.75)^n$ implies $n \geq \log_{0.75}(0.5) \approx 2.4$ (inequality sign changes direction as $0 < 0.75 < 1$). □

16. Let X be a random variable with binomial distribution $b(x;15,0.2)$. Find the probability that X lies within 2 standard deviations of its mean; i.e. compute $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$. To do this, make use of the table below.

n	k	$\binom{n}{k}$	0.01	0.05	0.10	0.15	0.20	0.25	0.30	1/3	0.35	0.40	0.45	0.49	0.50
15	0	1	0.8601	0.4633	0.2059	0.0874	0.0352	0.0134	0.0047	0.0023	0.0016	0.0005	0.0001	0.0000	
	1	15	0.1303	0.3658	0.3432	0.2312	0.1319	0.0668	0.0305	0.0171	0.0126	0.0047	0.0016	0.0006	0.0005
	2	105	0.0092	0.1348	0.2669	0.2856	0.2309	0.1559	0.0916	0.0599	0.0476	0.0219	0.0090	0.0040	0.0032
	3	455	0.0004	0.0307	0.1285	0.2184	0.2501	0.2252	0.1700	0.1299	0.1110	0.0634	0.0318	0.0166	0.0139
	4	1365		0.0049	0.0428	0.1156	0.1876	0.2252	0.2186	0.1948	0.1792	0.1268	0.0780	0.0478	0.0417
	5	3003		0.0006	0.0105	0.0449	0.1032	0.1651	0.2061	0.2143	0.2123	0.1859	0.1404	0.1010	0.0916
	6	5005		0.0000	0.0019	0.0132	0.0430	0.0917	0.1472	0.1786	0.1906	0.2066	0.1914	0.1617	0.1527
	7	6435			0.0003	0.0030	0.0138	0.0393	0.0811	0.1148	0.1319	0.1771	0.2013	0.1997	0.1964
	8	6435				0.0005	0.0035	0.0131	0.0348	0.0574	0.0710	0.1181	0.1647	0.1919	0.1964
	9	5005				0.0001	0.0007	0.0034	0.0116	0.0223	0.0298	0.0612	0.1048	0.1434	0.1527
	10	3003					0.0001	0.0007	0.0030	0.0067	0.0096	0.0245	0.0515	0.0827	0.0916
	11	1365						0.0001	0.0006	0.0015	0.0024	0.0074	0.0191	0.0361	0.0417
	12	455							0.0001	0.0003	0.0004	0.0016	0.0052	0.0116	0.0139
	13	105									0.0001	0.0003	0.0010	0.0026	0.0032
	14	15											0.0001	0.0004	0.0005
	15	1													
n	k	$\binom{n}{k}$	0.01	0.05	0.10	0.15	0.20	0.25	0.30	1/3	0.35	0.40	0.45	0.49	0.50

Solution. A random variable with binomial distribution $b(x; n, \theta)$ has mean $\mu = n\theta$ and standard deviation $\sigma = \sqrt{n\theta(1-\theta)}$. In this case

$$\mu = (15)(0.2) = 3$$

and

$$\sigma = \sqrt{(15)(0.2)(0.8)} = \sqrt{2.4} \approx 1.5492.$$

Thus (using the table values)

$$\begin{aligned} P(\mu - 2\sigma \le X \le \mu + 2\sigma) &= P(3 - 2\sqrt{2.4} \le X \le 3 + 2\sqrt{2.4}) \\ &= P(-0.0984 \le X \le 6.0984) \\ &= P(0 \le X \le 6) \\ &= \sum_{x=0}^{6} b(x; 15, 0.2) \\ &= 0.0352 + 0.1319 + 0.2309 + 0.2501 + 0.1876 + 0.1032 + 0.0430 \\ &= 0.9819 \end{aligned}$$

$\square$