 Expected Value III 2023-03-07

(some examples, one of a pitfall)

A pitfall: not every probability distribution has an expected value...

eg Consider the continuous random variable X with density function $f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & x < 1 \end{cases} = \begin{cases} x^{-2} & x \geq 1 \\ 0 & x < 1. \end{cases}$

Check it's a valid density:

1) $\frac{1}{x^2} \geq 0$ for all $x \geq 0$ & $0 \geq 0$ for all $x < 0$, so $f(x) \geq 0$ for all x. ✓

2) $\int_{-\infty}^{\infty} f(x)\,dx = \cancel{\int_{-\infty}^{1} 0\,dx} + \int_{1}^{\infty} x^{-2}\,dx = \lim_{a \to \infty} \int_{1}^{a} x^{-2}\,dx = \lim_{a \to \infty} \left.\frac{x^{-1}}{-1}\right|_{1}^{a}$

$= \lim_{a \to \infty} \left.\frac{-1}{x}\right|_{1}^{a} = \lim_{a \to \infty} \left(\left(\frac{-1}{a}\right) - \left(\frac{-1}{1}\right)\right) = \lim_{a \to \infty} \left(-\overset{\to 0}{\frac{1}{a}} + 1\right)$

$= -0 + 1 = 1$ ✓

... but it has no well-defined expected value:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{1} x \cdot 0\, dx + \int_{1}^{\infty} x \cdot \frac{1}{x^2}\, dx = \int_{-\infty}^{1} 0\, dx + \int_{1}^{\infty} \frac{1}{x}\, dx$$

$$= \lim_{a \to \infty} \int_{1}^{a} \frac{1}{x}\, dx = \lim_{a \to \infty} \ln(x) \Big|_{1}^{a} = \lim_{a \to \infty} \big(\ln(a) - \underbrace{\ln(1)}_{=0}\big)$$

$$= \lim_{a \to \infty} \ln(a) = \infty$$

... so there is no real number as our expected value.

eg A discrete example that's very similar to the above is the discrete random variable Y with probability distribution function $g(y) = \frac{6}{\pi^2 y^2}$. $\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \right)$

$(y = 1, 2, 3, \ldots)$

Then $E(Y) = \sum_{y=1}^{\infty} \frac{6}{\pi^2 y^2} \cdot y = \sum_{y=1}^{\infty} \frac{6}{\pi^2 y} = \frac{6}{\pi^2} \left(\sum_{y=1}^{\infty} \frac{1}{y} \right)$

diverges (adds up to ∞ too)

... so this has no well-defined expected value either.

Recall that if $g(x)$ is a function $\mathbb{R} \to \mathbb{R}$ and X is a random ~~value~~ variable, with probability dist./density fn. $f(x)$, then the expected value of $Y = g(X)$, is given by

$$E(Y) = \sum_x g(x)f(x) = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

$\overset{\shortparallel}{g(X)}$ (if X is discrete) (if X is continuous)

es X is a continuous random variable with density function $f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$, & $g(x) = x^2$.

$$E(X^2) = \int_{-\infty}^{\infty} g(x)f(x)\,dx = \underbrace{\cancel{\int_{-\infty}^{0} x^2 \cdot 0\,dx}}_{0} + \int_0^{\infty} x^2 e^{-x}\,dx$$

Use integration by parts with $u = x^2$, $v' = e^{-x}$, $u' = 2x$, $v = (-1)e^{-x}$

$$= \lim_{a\to\infty} \int_0^a x^2 e^{-x}\,dx = \lim_{a\to\infty}\left[-x^2 e^{-x}\Big|_0^a + \int_0^a 2x(-1)e^{-x}\,dx \right]$$

$$= \lim_{a\to\infty}\left[(-a^2 e^{-a}) - \cancel{(-0^2 e^{-0})} + 2\int_0^a x e^{-x}\,dx \right]$$

Parts again: $u = x$, $v' = e^{-x}$, $u' = 1$, $v = (-1)e^{-x}$

④

$$= \lim_{a\to\infty} \left[-\frac{a^2}{e^a} + 2\left(-xe^{-x}\Big|_0^a + \int_0^a 1\cdot(+1)e^{-x}\,dx \right) \right]$$

$$= \lim_{a\to\infty} \left[-\frac{a^2}{e^a} + 2\left((-ae^{-a}) - \cancel{(-0e^{-0})} + \int_0^a e^{-x}\,dx \right) \right]$$

$$= \lim_{a\to\infty} \left[-\frac{a^2}{e^a} + 2\left(-\frac{a}{e^a} + (-1)e^{-x}\Big|_0^a \right) \right]$$

$$= \lim_{a\to\infty} \left[-\frac{a^2}{e^a} + 2\left(-\frac{a}{e^a} + (-e^{-a}) + (+\underbrace{e^{-0}}_{1}) \right) \right]$$

$$= \lim_{a\to\infty} \left[-\frac{a^2}{e^a} - \frac{2a}{e^a} - \frac{2}{e^a} + 2 \right]$$

$$= \lim_{a\to\infty} \left[- \boxed{\frac{a^2+2a+2 \to \infty}{e^a \to \infty}} + 2 \right]$$

but exponential growth dominates polynomial growth,

so $\to 0$

$$= -0 + 2 = 2 \quad \text{ie} \quad E(X^2) = 2.$$

[This is an example of a moment (§4.5 in the notes)
$E(X^r)$ is the r^{th} moment of the random variable X.]

If X & Y are jointly distributed, we can (try to) compute the expected value of $\underbrace{g(X,Y)}_{Z}$ for any function of two variable $g(x,y)$.

__Def'n:__ (discrete case) $E(g(X,Y)) = \sum_x \sum_y g(x,y) \cdot f(x,y)$

(where $f(x,y)$ is the joint distribution function of X & Y)

__eg__ $g(x,y) = x+y$ $f(x,y)$ is given by

$y \backslash x$	1	2	3
0	0.2	0	0.1
1	0.1	0.2	0
2	0.1	0.1	0.2

$$E(X+Y) = \sum_{x=1}^{3} \sum_{y=0}^{2} (x+y) \cdot f(x,y)$$

$$\begin{aligned} &= (1+0)\cdot 0.2 + (2+0)\cdot 0 + (3+0)\cdot 0.1 \\ &\quad + (1+1)\cdot 0.1 + (2+1)\cdot 0.2 + (3+1)\cdot 0 \\ &\quad + (1+2)\cdot 0.1 + (2+2)\cdot 0.1 + (3+2)\cdot 0.2 \end{aligned}$$

$$\begin{aligned} &= 0.2 + 0 + 0.3 \\ &\quad + 0.2 + 0.6 + 0 \\ &\quad + 0.3 + 0.4 + 1.0 \end{aligned} = 3$$

__Def'n__ (cts. case)

$$E(g(x,y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) \, dy \, dx$$