MATH 1550H Expected Value II

2023-03-06

(for continuous random variables, plus some general properties)

Q.: If X is a continuous random variable, what value should we expect out of it on average?

Defn: Suppose X is a continous random variable with probability density function $f(x)$. Then the expected value of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx .$$

(Again, this is a kind of weighted average.)

eg. Suppose X has a uniform distribution on $[1,3]$.

ie $f(x) = \begin{cases} \frac{1}{3-1} & x \in [1,3] \\ 0 & x \notin [1,3] \end{cases} = \begin{cases} \frac{1}{2} & x \in [1,3] \\ 0 & x \notin [1,3] \end{cases}$

What is the expected value?

②

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{1} x\cdot 0\,dx + \int_{1}^{3} x\cdot\tfrac{1}{2}\,dx + \int_{3}^{\infty} \cancel{x}\cdot 0\,dx$$

$$= \cancel{\int_{-\infty}^{1} 0\,dx} + \tfrac{1}{2}\int_{1}^{3} x\,dx + \cancel{\int_{3}^{\infty} 0\,dx} = \tfrac{1}{2}\cdot\tfrac{x^2}{2}\Big|_1^3$$

$$= \tfrac{1}{2}\cdot\tfrac{9}{2} - \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{9}{4} - \tfrac{1}{4} = \tfrac{8}{4} = 2$$

Note that 2 is the mid-point of $[1,3]$.

* In general, if X has a uniform distribution on $[a,b]$, then $E(X) = \frac{a+b}{2}$.

Ex Suppose W has the density function $f(w) = \begin{cases} 3e^{-3w} & w \geq 0 \\ 0 & w < 0. \end{cases}$

[~~expontio~~ exponential distribution with $\lambda = 3$]

Then $E(W) = \int_{-\infty}^{\infty} w f(w)\,dw = \cancel{\int_{-\infty}^{0} w\cdot 0\,dw} + \int_{0}^{\infty} w\cdot 3e^{-3w}\,dw$

$$= \lim_{a\to\infty} \int_{0}^{a} 3we^{-3w}\,dw$$

But $\int_0^a \underbrace{3 w e^{-3w} \, dw}_{(-1)dx}$ Substitution $\quad x = -3w \quad w = -\frac{x}{3}$ ③

$dx = -3\,dw$

$(-1)dx = 3\,dw$

x	w
0	0
a	$-\frac{a}{3}$

$$= \int_0^{-a/3} \left(\cancel{-}\frac{x}{3}\right) e^x \, (\cancel{-}1)\,dx = \frac{1}{3}\int_0^{-a/3} x e^x \, dx$$

Integration by parts

$u = x \qquad v' = e^x$

$u' = 1 \qquad v = e^x$

$$= \frac{1}{3}\left[x e^x \Big|_0^{-a/3} - \int_0^{-a/3} 1 \cdot e^x \, dx \right]$$

$$= \frac{1}{3}\left[\left[\left(-\frac{a}{3} e^{-a/3}\right) - \left(0 \cdot e^0\right)\right] - e^x \Big|_0^{-a/3} \right]$$

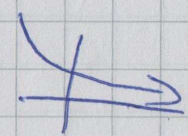
$$= \frac{1}{3}\left[-\frac{a}{3} e^{-a/3} - 0 - \left(e^{-a/3} - \underbrace{e^0}_{1}\right)\right]$$

$$= \frac{1}{3}\left[-\frac{a}{3} e^{-a/3} - e^{-a/3} + 1 \right]$$

... we plug this back into the limit

ix $\quad E(W) = \lim_{a\to\infty} \int_0^a 3w e^{-3w}\,dw$

$$= \lim_{a\to\infty} \frac{1}{3}\Big[\underbrace{\underbrace{-\tfrac{a}{3}}_{-\infty}\;\underbrace{e^{-a/3}}_{0}}_{0} - \underbrace{e^{-a/3}}_{0} + 1\Big]$$



... but exponentials go faster than powers of x, so $e^{-a/3} \to 0$ wins

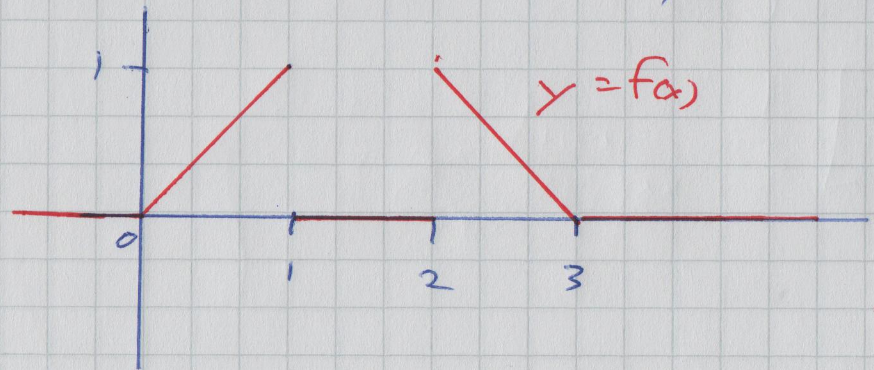
(Could use l'Hôpital's Rule to be more formal)

$$= \frac{1}{3}\big[0 - 0 + 1\big] = \frac{1}{3}$$

Note this is a value that W can actually take on, similarly to the last example.

However, this need not be the case for continuous distributions.

eg X has density function $f(x) = \begin{cases} x & x \in [0,1] \\ 3-x & x \in [2,3] \\ 0 & \text{otherwise} \end{cases}$



$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \cancel{\int_{-\infty}^{0} x \cdot 0\,dx} + \int_0^1 x \cdot x\,dx + \cancel{\int_1^2 x \cdot 0\,dx} + \int_2^3 (3-x)\,dx + \cancel{\int_3^{\infty} x \cdot 0\,dx}$$

$$= \int_0^1 x^2\,dx + \int_2^3 (3x - x^2)\,dx$$

$$= \frac{x^3}{3}\Big|_0^1 + \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_2^3$$

$$= \left[\frac{1}{3} - 0\right] + \left[\left(\frac{27}{2} - \frac{27}{3}\right) - \left(\frac{12}{2} - \frac{8}{3}\right)\right]$$

$$= \frac{1}{3} + \left[\frac{15}{2} - \frac{19}{3}\right] = \frac{2}{6} + \left[\frac{45}{6} - \frac{38}{6}\right]$$

$$= \frac{2}{6} + \frac{7}{6} = \frac{9}{6} = \frac{3}{2} = 1.5$$ ... which is a value X does not ever take on.

Some general properties of expected values:

In a lot of cases, random variables are defined in terms of other variables, ie $Y = g(X)$ for some function $g: \mathbb{R} \to \mathbb{R}$. It's a pain to work out $E(Y)$ from the defn's of ~~Y~~ Y & expected value, because working out the probability distribution/density of Y is extra work. Shortcut:

Defn: If X is a random variable & $Y = g(X)$, then $E(Y) = E(g(X))$

$$= \sum_x g(x) f(x) \quad [\text{discrete}]$$

$$= \int_{-\infty}^{\infty} g(x) f(x)\, dx \quad [\text{continuous}]$$

eg $X = \#$ heads in 3 tosses of a fair coin

$Y = 2X+1 \quad (\text{ie } g(x) = 2x+1)$

\# ways to pick the x tosses that get heads

probability of heads on the selected tosses

$$E(Y) = E(2X+1) = \sum_x g(x)f(x)$$

$$= \sum_{x=0}^{3} (2x+1)\,P(X=x) = \sum_{x=0}^{\cancel{2}\,3} (2x+1)\binom{3}{x}\underbrace{\left(\tfrac{1}{2}\right)^x\left(\tfrac{1}{2}\right)^{3-x}}_{\left(\frac{1}{2}\right)^3 = \frac{1}{8}}$$

probability of tails on the other toss

$$\begin{aligned}
&= (2\cdot 0+1)\cdot\binom{3}{0}\cdot\frac{1}{8} &&= 1\cdot 1\cdot\frac{1}{8}\\
&\quad+(2\cdot 1+1)\cdot\binom{3}{1}\cdot\frac{1}{8} &&\quad+ 3\cdot 3\cdot\frac{1}{8}\\
&\quad+(2\cdot 2+1)\cdot\binom{3}{2}\cdot\frac{1}{8} &&\quad+ 5\cdot 3\cdot\frac{1}{8}\\
&\quad+(2\cdot 3+1)\cdot\binom{3}{3}\cdot\frac{1}{8} &&\quad+ 7\cdot 1\cdot\frac{1}{8}
\end{aligned}$$

$$= \frac{1}{8} + \frac{9}{8} + \frac{15}{8} + \frac{7}{8} = \frac{32}{8} = 4$$

Note that $E(X) = \sum_{x=0}^{3} x\binom{3}{x}\cdot\frac{1}{8} = 0\cdot 1\cdot\frac{1}{8} + 1\cdot 3\cdot\frac{1}{8} + 2\cdot 3\cdot\frac{1}{8} + 3\cdot 1\cdot\frac{1}{8}$

$$= \cancel{8}\,0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$

Not by coincidence $E(2X+1) = 4 = \cancel{3}\,(2\cdot 1.5+1)$

$$= 2E(X)+1$$

Prop.: If X is a random variable, and a & b are constants, then

$$E(aX+b) = aE(X) + b.$$

proof: (continuous case – discrete case is in the notes)

Suppose X has density function $f(x)$.

Then
$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \big(ax\, f(x) + b f(x)\big)\,dx = \int_{-\infty}^{\infty} ax\, f(x)\,dx + \int_{-\infty}^{\infty} b f(x)\,dx$$

$$= a \underbrace{\int_{-\infty}^{\infty} x\, f(x)\,dx}_{E(X)} + b \underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{=1}$$

since $f(x)$ is a density function

$$= aE(X) + b.$$

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Q.: If X and Y are random variables, then is $E(X+Y) = E(X) + E(Y)$?

Yes, if X & Y are independent.

No, or at least maybe not, if X & Y are not independent. (It can happen anyway in rare cases.)