MATH 1550H Expected Value I

2023-03-06

(1)

(Ch. 4 in the notes) (for discrete random variables)

Q.: If X is a random variables, what number, on average, should we expect it to deliver?

Def'n: Suppose X is a discrete random variable with probability distribution function $f(x)$. Then the expected value of X is

$$E(X) = \sum_x x f(x) = \sum_x x P(X=x).$$

This is a kind of weighted average of the values of X; weighting by the probability they occur.

eg Toss a fair coin once & let X = # heads that occur. Possible values of X are 0 & 1; each with probability $\frac{1}{2}$.

$$E(X) = \sum_x x P(X=x) = 0 \cdot P(X=0) + 1 \cdot P(X=1) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

<u>Note:</u> The "expected value" of X might be a number (ie value) that X never delivers.

So what does it mean? It means that if we repeated the experiment many times we would expect an average value of $\frac{1}{2}$ (ie about the tosses would be heads).

<u>eg</u> Suppose the coin we toss once is biased, say
$P(H) = \frac{1}{3}$ & $P(T) = 1 - P(H) = 1 - \frac{1}{3} = \frac{2}{3}$.
X = # heads on the one toss. (Possible values: 0 & 1).

$$E(X) = \sum_x x P(X=x) = 0 \cdot P(X=0) + 1 \cdot P(X=1)$$
$$= 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3}.$$

e) Roll a fair standard die once.

Y = # on the face that comes up.

Possible values of Y are 1, 2, 3, 4, 5, 6, which occur with an equal probability of $\frac{1}{6}$ [Uniform distribution on 1, 2, ..., 6.]

$$E(Y) = \sum_y y P(Y=y) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6}$$

$$= \frac{1+2+3+4+5+6}{6}$$ [actual average of the possible outputs]

$$= \frac{21}{6} = \frac{7}{2} = 3.5$$

Once again the expected value is not a value that actually occurs.

eg Roll a fair standard die 4 times

Z = # times that either 3 or 6 came up in the four rolls.

Possible values: 0, 1, 2, 3, 4

Probabilities: $P(Z=0) = P(\text{You got } 1,2,4, \text{ or } 5 \text{ each time})$

$$= \binom{4}{0}\cdot\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

$P(Z=1) = P(\text{You got } 3 \text{ or } 6 \text{ once})$

$$= \binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 = \frac{32}{81}$$

$$P(Z=2) = \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{24}{81}$$

$$P(Z=3) = \binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1 = \frac{8}{81}$$

$$P(Z=4) = \binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0 = \frac{1}{81}$$

Later on we'll see a much more labour-saving way to compute this.

$$E(Z) = \sum_z z P(Z=z) = 0\cdot\frac{16}{81} + 1\cdot\frac{32}{81} + 2\cdot\frac{24}{81} + 3\cdot\frac{8}{81} + 4\cdot\frac{1}{81}$$

$$= \frac{0+32+48+24+4}{81} = \frac{108}{81} = \frac{12\cdot\cancel{9}}{9\cdot\cancel{9}} = \frac{4\cdot\cancel{3}}{3\cdot\cancel{3}} = \frac{4}{3}$$

$$\approx 1.3333$$

Again, it's not a value Z actually takes on.

ex Toss a fair coin until you get a tail.

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X = # tosses required

H HT HHT HHHT ...

Possible values of X are 1, 2, 3, 4, ...

Probabilities: $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$ $\frac{1}{16}$ ...

(In general, $f(x) = P(X=x) = \left(\frac{1}{2}\right)^x$.)

$$E(X) = \sum_x x f(x) = \sum_x x P(X=x) = \sum_{x=1}^{\infty} x \left(\frac{1}{2}\right)^x$$

$$= 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 4\cdot\frac{1}{16} + 5\cdot\frac{1}{32} + \cdots$$

$$= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots$$

$$= \frac{1}{2} + \left(\frac{1}{4}+\frac{1}{4}\right) + \left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right) + \left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}\right) + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2}\Big/\frac{1}{2} = \Big(\frac{1}{2}$$

$$+ \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \qquad + \frac{\frac{1}{4}}{1-\frac{1}{2}} \qquad + \frac{1}{4}\Big/\frac{1}{2} \qquad + \frac{1}{4}$$

$$+ \frac{1}{8} + \frac{1}{16} + \cdots \qquad + \frac{\frac{1}{8}}{1-\frac{1}{2}} \qquad + \frac{1}{8}\Big/\frac{1}{2} \qquad + \frac{1}{8}$$

$$+ \frac{1}{16} + \frac{1}{32} + \cdots \qquad + \frac{\frac{1}{16}}{1-\frac{1}{2}} \qquad + \frac{1}{16}\Big/\frac{1}{2} \qquad + \frac{1}{16}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \Big)\Big/\frac{1}{2}$$

geometric series with different starting points but all have the common ratio of $r = \frac{1}{2}$

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}$$

(as long as $|r| < 1$)

$$= 2\cdot\frac{\frac{1}{2}}{1-\frac{1}{2}} = 2\cdot\frac{1/2}{1/2} = 2$$

Note: In this case the expected value is a number the random variable might actually deliver.