TRENT UNIVERSITY, WINTER 2018

MATH 1550H Test #2

Friday, 16 March

Time: 50 minutes

Name: Solutions

STUDENT NUMBER: 0123456

Question	**Mark**	
1	____	
2	____	
3	____	
Total	____	/30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator, a standard normal table, and an aid sheet.

1. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. A fair standard die is rolled until either 1 or 2 comes up. What are the expected value $E(Y)$ and variance $V(Y)$ of the random variable Y that counts the number of rolls that occur in this experiment?

b. Verify that $f(x) = \begin{cases} \dfrac{2x}{(1+x^2)^2} & x \geq 0 \\ 0 & x \leq 0 \end{cases}$ is a valid density function.

c. A fair coin is tossed three times. The random variable U counts how many tails came up in the three tosses and the random variable V counts how many heads came up on the second of the three tosses. Determine whether U and V are independent or not.

SOLUTIONS. **a.** Each face of a fair standard die has a probability of $\frac{1}{6}$ of coming up on any given roll, so the probability that a roll will come up with 1 or 2 is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. Rolling the die until it comes up with 1 or 2 and counting the number of rolls required therefore has a geometric distribution with probability of success $p = \frac{1}{3}$ and of failure of $q = 1 - p = \frac{2}{3}$ on each triel. It follows that $E(Y) = \frac{1}{p} = \frac{1}{1/3} = 3$ and $V(Y) = \frac{q}{p^2} = \frac{2/3}{(1/3)^2} = \frac{2}{3} \cdot 3^2 = 6$. $\square$

b. First, when $x \leq 0$, $f(x) = 0 \geq 0$, and when $x \geq 0$, $f(x) = \frac{2x}{(1+x^2)^2} \geq 0$ because $2x \geq 0$ and $\left(1 + x^2\right)^2 \geq 1 > 0$. Thus $f(x) \geq 0$ for all x, as required.

Second, we need to check that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. We will make use of the substitution $u = 1 + x^2$, so $du = 2x\,dx$. Note that when $x = 0$, $u = 1 + 0^2 = 1$, and as $x \to \infty$, $u \to \infty$. Then :

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} \frac{2x}{(1+x^2)^2}\,dx = 0 + \int_{1}^{\infty} \frac{1}{u^2}\,du \\ &= \left.\frac{-1}{u}\right|_1^{\infty} = \left(\frac{-1}{\infty}\right) - \left(\frac{-1}{1}\right) = (-0) - (-1) = 1\end{aligned}$$

Hence $f(x)$ saitisfies both parts of the definition for a valid density function. $\square$

c. U and V are not independent. Note that $P(V = 1) = P(H) = \frac{1}{2}$ while $P(U = 3) = P(TTT) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$. However, if $U = 3$, the second toss must have been a tail, making $V = 1$ impossible, so $P(U = 3 \;\&\; V = 1) = 0$. Since $P(U = 3 \;\&\; V = 1) = 0 \neq \frac{1}{16} = \frac{1}{8} \cdot \frac{1}{2} = P(U = 3)P(V = 1)$, U and V cannot be independent. $\blacksquare$

2. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Find the expected value $E(W)$ and variance $V(W)$ of the continuous random variable W that has as its density function $h(w) = \begin{cases} \frac{3}{4}\left(1-w^2\right) & -1 \le w \le 1 \\ 0 & \text{otherwise} \end{cases}$.

b. A fair four-sided die with faces numbered 0, 2, 3, and 5, respectively, is rolled once. What are the expected value and variance of the number that comes up?

c. Suppose that the continuous random variable Z has a standard normal distribution. Find $P(Z > 1.26)$.

Solutions. **a.** Here we go:

$$\begin{aligned}
E(W) &= \int_{-\infty}^{\infty} w \cdot h(w)\,dw = \int_{-\infty}^{-1} w \cdot 0\,dw + \int_{-1}^{1} w \cdot \frac{3}{4}\left(1-w^2\right)\,dw + \int_{1}^{\infty} w \cdot 0\,dw \\
&= 0 + \frac{3}{4}\int_{-1}^{1}\left(w - w^3\right)\,dw + 0 = \frac{3}{4}\left(\frac{w^2}{2} - \frac{w^4}{4}\right)\Big|_{-1}^{1} \\
&= \frac{3}{4}\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \frac{3}{4}\left(\frac{(-1)^2}{2} - \frac{(-1)^4}{4}\right) = \frac{3}{4}\left(\frac{1}{2} - \frac{1}{4}\right) - \frac{3}{4}\left(\frac{1}{2} - \frac{1}{4}\right) \\
&= \frac{3}{4}\cdot\frac{1}{4} - \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{16} - \frac{3}{16} = 0 \quad \text{One down!} \\
E\left(W^2\right) &= \int_{-\infty}^{\infty} w^2 \cdot h(w)\,dw = \int_{-\infty}^{-1} w^2 \cdot 0\,dw + \int_{-1}^{1} w^2 \cdot \frac{3}{4}\left(1-w^2\right)\,dw + \int_{1}^{\infty} w^2 \cdot 0\,dw \\
&= 0 + \frac{3}{4}\int_{-1}^{1}\left(w^2 - w^4\right)\,dw + 0 = \frac{3}{4}\left(\frac{w^3}{3} - \frac{w^5}{5}\right)\Big|_{-1}^{1} \\
&= \frac{3}{4}\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \frac{3}{4}\left(\frac{(-1)^3}{3} - \frac{(-1)^5}{5}\right) = \frac{3}{4}\left(\frac{1}{3} - \frac{1}{5}\right) - \frac{3}{4}\left(\frac{-1}{3} - \frac{-1}{5}\right) \\
&= \frac{3}{4}\cdot\frac{2}{15} - \frac{3}{4}\cdot\frac{-2}{15} = \frac{1}{10} - \frac{-1}{10} = \frac{2}{10} = \frac{1}{5}
\end{aligned}$$

Thus $V(W) = E\left(W^2\right) - [E(W)]^2 = \frac{1}{5} - 0^2 = \frac{1}{5}$. □

b. Each face on a fair die has an equal chance of coming up. Since the given die has four faces, this equal probability is $\frac{1}{4} = 0.25$. If X is the random variable that gives the number on the face that comes up, then the expected value is $E(X) = \sum_k k \cdot P(X = k) = 0 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 5 \cdot \frac{1}{4} = \frac{10}{4} = \frac{5}{2} = 2.5$. Similarly, $E\left(X^2\right) = \sum_k k^2 \cdot P(X = k) = 0^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{1}{4} + 5^2 \cdot \frac{1}{4} = \frac{38}{4} = \frac{19}{2} = 9.5$. Then $V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{19}{2} - \left(\frac{5}{2}\right)^2 = \frac{38}{4} - \frac{25}{4} = \frac{13}{4} = 3.25$. □

c. Note that $P(Z > 1.26) = 1 - P(Z \le 1.26)$. Consulting our cumulative standard normal table, the entry in the row for $z = 1.2$ and the column for 0.06 gives $P(Z \le 1.26) \approx 0.8962$. It follows that $P(Z > 1.26) = 1 - P(Z \le 1.26) \approx 1 - 0.8962 = 0.1038$. □

c. *(Alternate)* By the symmetry of the density function of the standard normal distribution about $z = 0$, $P(Z > 1.26) = P(Z < -1.26)$. Consulting our cumulative standard normal table, the entry in the row for $z = -1.2$ and the column for 0.06 gives $P(Z \le 1.26) \approx 0.1038$. Thus $P(Z > 1.26) = P(Z < -1.26) \approx 0.1038$. ■

3. Do *one (1)* of **a** or **b**. *[10]*

a. A fair coin is tossed until it comes up tails. This experiment is repeated independently three times, with the random variables X_1, X_2, and X_3 recording the number of tosses on the first, second, and third run of the experiment, respectively. Find the expected value and variance of $X = X_1 + X_2 + X_3$, as well as the probability function of X.

b. The continuous random variable Y has density function $g(y) = \begin{cases} ye^{-y} & y \geq 0 \\ 0 & y \leq 0 \end{cases}$. Show that $g(y)$ is a valid density function and find the expected value of Y.

SOLUTIONS. **a.** Observe that $X = X_1 + X_2 + X_3$ counts the number of tosses of a fair coin required for a third head to come up. Thus X has a negative binomial distribution with $p = \frac{1}{2}$, $q = 1 - \frac{1}{2} = \frac{1}{2}$, and $k = 3$ successes. It follows that the probability function of X is given by $m(x) = P(k\text{th success on } x\text{th trial}) = \binom{x-1}{k-1} q^{x-k} p^k = \binom{x-1}{3-1} \left(\frac{1}{2}\right)^{x-3} \left(\frac{1}{2}\right)^3 = \binom{x-1}{2} \left(\frac{1}{2}\right)^x$, when x is an integer ≥ 3, and $m(x) = 0$ otherwise. It also follows that $E(X) = \frac{k}{p} = \frac{3}{1/2} = 6$ and $V(X) = \frac{kq}{p^2} = \frac{3 \cdot (1/2)}{(1/2)^2} = \frac{3}{1/2} = 6$. $\square$

b. First, observe that when $y \leq 0$, $g(y) = 0 \geq 0$, and when $y \geq 0$, $g(y) = ye^{-y} \geq 0$ because $y \geq 0$ and $e^{-y} > 0$. Second, we check that $\int_{-\infty}^{\infty} g(y)\, dy = 1$. We will use intregration by parts with $u = y$ and $v' = e^{-y}$, so that $u' = 1$ and $v = (-1)e^{-y}$. [The fact that $\int e^{-y}\, dy = (-1)e^{-y}$ is left to the alert reader.]

$$\begin{aligned} \int_{-\infty}^{\infty} g(y)\, dy &= \int_{-\infty}^{0} 0\, dy + \int_{0}^{\infty} ye^{-y}\, dy = 0 + y \cdot (-1)e^{-y}\Big|_0^\infty - \int_0^\infty 1 \cdot (-1)e^{-y}\, dy \\ &= \left[-ye^{-y}\right]\Big|_0^\infty + \int_0^\infty e^{-y}\, dy = \left[-\infty \cdot e^{-\infty}\right] - \left[0e^{-0}\right] + (-1)e^{-y}\Big|_0^\infty \\ &= 0 - 0 + (-1)e^{-\infty} - (-1)e^{-0} = -0 + 1 = 1 \quad [e^{-y} \to 0 \text{ beats } y \to \infty.] \end{aligned}$$

Thus $g(y)$ satisfies both conditions to be a valid density function. Third, we compute $E(Y)$. We will use parts again, this time with $u = y^2$ and $v' = e^{-y}$, so $u' = 2y$ and $v = (-1)e^{-y}$; we will also use the part of the calculation above that showed that $\int_0^\infty ye^{-y}\, dy = 1$.

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} y \cdot g(y)\, dy = \int_{-\infty}^{0} y \cdot 0\, dy + \int_0^\infty y \cdot ye^{-y}\, dy = \int_{-\infty}^{0} 0\, dy + \int_0^\infty y^2 e^{-y}\, dy \\ &= 0 + y^2 \cdot (-1)e^{-y}\Big|_0^\infty - \int_0^\infty 2y \cdot (-1)e^{-y}\, dy = \left[-y^2 e^{-y}\right]\Big|_0^\infty + 2\int_0^\infty ye^{-y}\, dy \\ &= \left[-\infty^2 e^{-\infty}\right] - \left[-0^2 e^{-y}\right] + 2 \cdot 1 \quad \text{(by the previous calculation)} \\ &= 0 - 0 + 2 = 2 \quad [e^{-y} \to 0 \text{ beats } y^2 \to \infty.] \qquad \blacksquare \end{aligned}$$

[Total = 30]