Trent University, Winter 2018

MATH 1550H Test #1

Friday, 9 February

Time: 50 minutes

Name: Solutions

Student Number: 0123456

Question	Mark	
1	____	
2	____	
3	____	
Total	____	/30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. A fair coin is tossed five times. What is the probability that at least four tails will come up?

b. Compute $P(X \geq 1)$ if the random variable X has the probability density function

$$f(x) = \begin{cases} \frac{1}{2}(1-x) & -1 \leq x \leq 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases} .$$

c. Suppose that A and B are events in some sample space, with $P(A) = P(B) = \frac{1}{2}$ and $P(A \cup B) = \frac{2}{3}$. What is $P(A \cap B)$?

SOLUTIONS. **a.** Suppose A is the event that at least four tails come up in five tosses of a fair coin. Then $A = \{HTTTT, THTTT, TTHTT, TTTHT, TTTTH, TTTTT\}$, so, since any sequence ω of five tosses of a fair coin has probability $m(\omega) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$,

$$\begin{aligned} P(A) &= m(HTTTT) + m(THTTT) + m(TTHTT) \\ &\quad + m(TTTHT) + m(TTTTH) + m(TTTTT) \\ &= \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} = \frac{6}{32} = \frac{3}{16} = 0.1875 . \quad \square \end{aligned}$$

b. *(With calculus.)* $f(x) = 0$ for $x \geq 1$, so $P(X \geq 1) = \int_1^\infty f(x)\,dx = \int_1^\infty 0\,dx = 0.$ $\square$

b. *(Without calculus.)* $f(x) = 0$ for $x \geq 1$, so $P(X \geq 1)$, which is the area under the graph of $y = f(x)$ for $x \geq 1$, is 0. $\square$

c. Rearranging $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to solve for $P(A \cap B)$ tells us that $P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{2}{3} = 1 - \frac{2}{3} = \frac{1}{3}$. $\blacksquare$

2. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. A fair coin is tossed, and then tossed some more until it comes up with the face other than the one that came up on the first toss. What are the sample space and probability function for this experiment?

b. A hand of five cards is drawn randomly and simultaneously from a standard 52-card deck. How many such hands have three ♠s and two ♢s?

c. A fair coin is tossed until it comes up heads. Let A be the event that at least 5 tosses are required, and let B be the event that at least 3 tosses are required. Compute $P(A|B)$.

SOLUTIONS. **a.** It is not hard to see that the outcomes of this experiment are sequences that are all heads or all tails until the last flip, which is the other side. Thus the sample space is $\Omega = \{ HT, TH, HHT, TTH, HHHT, TTTH, \dots \}$; in general, the sequences of coin tosses in Ω look like H^kT or T^kH for some $k \geq 1$.

Since we are tossing a fair coin, the probability of each outcome is simply $\left(\frac{1}{2}\right)^n$, where n is the number of tosses in the outcome. Thus the probability function is given by $m\left(H^kT\right) = m\left(T^kH\right) = \left(\frac{1}{2}\right)^{k+1}$. □

b. There are thirteen ♠s and thirteen ♢s in a standard deck. Since there are $\binom{13}{3} = 286$ ways to choose three out of thirteen ♠s and $\binom{13}{2} = 78$ ways to choose two out of thirteen ♢s, there are $\binom{13}{3}\binom{13}{2} = 286 \cdot 78 = 22308$ five-card hands that have three ♠s and two ♢s. (Note that a hand with three ♠s and two ♢s has five cards already.) □

c. Our sample space is $\Omega = \{ H, TH, TTH, TTTH, TTTTH, \dots \}$ and the probability function is given by $m(\omega) = \left(\frac{1}{2}\right)^n$, where n is the number of tosses in the outcome. Then $A = \{TTTTH, TTTTTH, TTTTTTH \dots\}$ and $B = \{TTH, TTTH, TTTTH, \dots\}$; note that $A \cap B = A$ since $A \subseteq B$. We compute $P(A)$ and $P(B)$ indirectly:

$$\begin{aligned}
P(A) = 1 - P\left(\bar{A}\right) &= 1 - [m(H) + m(TH) + m(TTH) + m(TTTH)] \\
&= 1 - \left[\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4\right] \\
&= 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}\right] = 1 - \frac{15}{16} = \frac{1}{16} \\
P(B) = 1 - P\left(\bar{B}\right) &= 1 - [m(H) + m(TH)] = 1 - \left[\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2\right] \\
&= 1 - \left[\frac{1}{2} + \frac{1}{4}\right] = 1 - \frac{3}{4} = \frac{1}{4}
\end{aligned}$$

It follows that $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)}{P(B)} = \dfrac{1/16}{1/4} = \dfrac{1}{16} \cdot \dfrac{4}{1} = \dfrac{4}{16} = \dfrac{1}{4} = 0.25$. ■

3. Do *one (1)* of **a** or **b**. *[10]*

a. If you were to pick an answer to this question at random from among the choices below, what is the probability that it would be correct? Explain your answer!

(1) 0.2 (2) $1/\pi$ (3) $1/5$ (4) 0.0 (5) $4/10$

b. What is the probability that a six was rolled exactly once in three rolls of a fair standard die, given that you know that a five was rolled exactly once?

SOLUTIONS. **a.** *Abandon all hope, ye who answered this! For there is no right answer:* Choosing among the five choices at random, you would have a probability of $1/5 = 0.2$ of choosing each one. Since choices (1) and (3) are 0.2 and $1/5$, respectively, this value has a probability of $2/5 = 0.4$ of being chosen, so cannot be correct. Choice (5), $4/10$, cannot be correct, since it has only a $1/5 = 0.2$ chance of being chosen. Choice (2) cannot be correct, since the probability of choosing any combination of the five alternatives must be an integer multiple of $1/5$, and $1/\pi$ is not an integer multiple of $1/5$. You might think this leaves a probability of 0 of selecting a correct alternative, but 0 is alternative (4). which has a probability of $0.2 \neq 0$ of being chosen at random ... *The correct solution here is that there is no way to answer the given question, because it is internally contradictory.* $\square$

b. A fair standard die has six faces, numbered one through six, that each have a probability of $\frac{1}{6}$ of coming up when the die is rolled. Each sequence – order matters here – of three rolls of a fair standard die then has a probability of $\left(\frac{1}{6}\right)^3 = \frac{1}{216} \approx 0.0046$. Let A be the event that a six was rolled exactly once in three rolls of the die, and let B be the event that a five was rolled exactly once in three rolls of the die.

We first compute the probability of B. There are $\binom{3}{1} = 3$ ways to choose the roll in which a five occurs. The probability that a five occurs on that roll is $\frac{1}{6}$, and the probability that something other than a five occurs on each of the other two rolls is $\frac{5}{6}$. It follows that $P(B) = \binom{3}{1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2 = 3 \cdot \frac{1}{6} \cdot \frac{25}{36} = \frac{75}{216} = \frac{25}{72} \approx 0.3472$. (A similar analysis will show that $P(A) = \frac{25}{72}$, too.)

We next compute the probability of $A \cap B$, the event that exactly one six and exactly one five were rolled in three rolls of the die. There are $\binom{3}{1} = 3$ ways to choose the roll in which a six occurs, and then $\binom{2}{1} = 2$ ways to choose the roll – among the two rolls remaining – in which a five occurs. Each of these rolls has a probability of $\frac{1}{6}$ of being filled with a six and a five respectively; the remaining roll has a probability of $\frac{4}{6}$ of having something other than a five or six come up. It follows that $P(A \cap B) = \binom{3}{1} \cdot \binom{2}{1} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{4}{6} = 3 \cdot 2 \cdot \frac{4}{216} = \frac{24}{216} = \frac{1}{9} \approx 0.1111$.

Thus $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/9}{25/72} = \dfrac{1}{9} \cdot \dfrac{72}{25} = \dfrac{8}{25} = 0.32$. ■

[Total = 30]