Mathematics 1550H – Introduction to probability
Trent University, Winter 2018

Solutions to Assignment # 9
The bell curve strikes again!

A random variable X has a *normal distribution with mean* μ *and standard deviation* σ if it has as its density function

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} .$$

Notes for the solutions. The solutions given below depend on recollecting or working out the values of certain integrals for the standard normal distribution. Recall that if the random variable Z has the standard normal distribution has density function $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{z^2/2}$. We worked out in class that this was a valid probability density, including the fact that:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{z^2/2}\, dz = \int_{-\infty}^{\infty} \varphi(z)\, dz = 1$$

Since the standard normal distribution has expected value 0, we must have that:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ze^{z^2/2}\, dz = \int_{-\infty}^{\infty} z\varphi(z)\, dz = E(Z) = 0$$

(This integral is actually quite easy to solve using the substitution $u = z^2/2$.) Since the standard normal distribution also has standard deviation 1, it must have variance $1^2 = 1$, and as $V(Z) = E\left(Z^2\right) - [E(Z)]^2$, it follows that:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{z^2/2}\, dz = \int_{-\infty}^{\infty} z^2\varphi(z)\, dz = E\left(Z^2\right) = V(Z) - E(Z) = 1 - 0^2 = 1$$

Our strategy in the problems below will be to transform and decompose the integrals for the given normal distribution into the integrals above that arise in the standard normal distribution and then simply plug in their values. We will use the substitution $z = \dfrac{x-\mu}{\sigma}$ in both problems; then $x = \sigma z + \mu$, so $dx = \sigma\, dz$ and $dz = \dfrac{1}{\sigma}\, dx$, and, since $\sigma > 0$, we have $\begin{array}{ccc} x & -\infty & \infty \\ z & -\infty & \infty \end{array}$ when we change the limits.

1. Show that $E(X) = \mu$. *[5]*

Solution. Set up the integral for $E(X)$, substitute as above, and chug away:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x\varphi(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-z^2/2}\, dz \\ &= \sigma \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ze^{z^2/2}\, dz + \mu \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{z^2/2}\, dz = \sigma \cdot 0 + \mu \cdot 1 = \mu \quad \blacksquare \end{aligned}$$

2. Show that $\sigma = \sqrt{V(X)}$. *[5]*

Solution. Set up the integral for $E\left(X^2\right)$, substitute as above, and chug away:

$$\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 \varphi(x)\, dx = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu)^2 e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sigma^2 z^2 + 2\sigma\mu z + \mu^2\right) e^{-z^2/2}\, dz \\
&= \sigma^2 \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{z^2/2}\, dz + 2\sigma\mu \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{z^2/2}\, dz + \mu^2 \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{z^2/2}\, dz \\
&= \sigma^2 \cdot 1 + 2\sigma\mu \cdot 0 + \mu^2 \cdot 1 = \sigma^2 + \mu^2
\end{aligned}$$

From this it follows that $V(X) = E\left(X^2\right) - [E(X)]^2 = \left(\sigma^2 + \mu^2\right) - \mu^2 = \sigma^2$, and so the variance of X is $\sqrt{V(X)} = \sqrt{\sigma^2} = \sigma$. ■