Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 2018

Solutions to Assignment # 8
Convolution!

Suppose that the independent continuous random variables X and Y both have an exponential distribution with $\lambda = 1$. Let $U = X + Y$.

1. Find the expected value $E(U)$ and variance $V(U)$. *[2]*

SOLUTION. An exponential distribution with parameter $\lambda = 1$ has expected value $\mu = \dfrac{1}{\lambda} = \dfrac{1}{1} = 1$ and variance $\sigma^2 = \dfrac{1}{\lambda^2} = \dfrac{1}{1^2} = 1$. It follows that $E(U) = E(X+Y) = E(X) + E(Y) = 1 + 1 = 2$. As X and Y are independent, we also get that $V(U) = V(X+Y) = V(X) + V(Y) = 1 + 1 = 2$. $\square$

2. Find the density function of U. *[8]*

SOLUTION. The density finction of an exponential distribution with parameter $\lambda = 1$ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & t \geq 0 \\ 0 & t < 0 \end{cases} = \begin{cases} 1e^{-1x} & t \geq 0 \\ 0 & t < 0 \end{cases} = \begin{cases} e^{-x} & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

To find the density function of $U = X + Y$, we compute the convolution of $f(x)$ with itself:

$$\begin{aligned}
(f * f)(u) &= \int_{-\infty}^{\infty} f(x) f(u-x)\, dx = \int_{-\infty}^{0} 0 \cdot f(u-x)\, dx + \int_{0}^{\infty} e^{-x} f(u-x)\, dx \\
&= 0 + \begin{cases} \int_0^u e^{-x} e^{-(u-x)}\, dx + \int_u^\infty e^{-x} \cdot 0\, dx & \text{if } u \geq 0 \\ \int_0^\infty e^{-x} \cdot 0\, dx & \text{if } u < 0 \end{cases} \\
&\quad \text{(Since } f(u-x) = e^{-(u-x)} = e^{x-u} \text{ when } u - x \geq 0, \textit{ i.e.} \text{ when } x \leq u, \\
&\quad \text{and } f(u-x) = 0 \text{ otherwise.)} \\
&= \begin{cases} \int_0^u e^{-x} e^{x} e^{-u}\, dx + \int_u^\infty 0\, dx & \text{if } u \geq 0 \\ \int_0^\infty 0\, dx & \text{if } u < 0 \end{cases} \\
&= \begin{cases} \int_0^u e^{-u}\, dx + 0 & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases} = \begin{cases} e^{u} \int_0^u dx + 0 & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases} \\
&\quad \text{(Since } e^{-u} \text{ is a constant as far as } x \text{ is concerned.)} \\
&= \begin{cases} e^{-u} x\big|_0^u & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases} = \begin{cases} e^{-u} \cdot u - e^{-u} \cdot 0 & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases} \\
&= \begin{cases} ue^{-u} - 0 & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases} = \begin{cases} ue^{-u} & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}
\end{aligned}$$

Thus the density function of $U = X + Y$ is $(f * f)(u) = \begin{cases} ue^{-u} & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}$. $\blacksquare$