Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 2018

Solutions to Assignment # 7
An expected value and variance

Consider the random variable X with density function $f(x) = \begin{cases} 0 & x < 1 \\ 2x^{-3} & x \geq 1 \end{cases}$.

1. Determine whether $f(x)$ is a valid probability density. *[4]*

SOLUTION. First, when $x < 1$, $f(x) = 0 \geq 0$, and when $x \geq 1$, $f(x) = 2x^{-3} = \frac{2}{x^3} \geq 0$ because $x^3 > 0$ when $x \geq 1 > 0$. Thus $f(x) \geq 0$ for all $x \in \mathbb{R}$.

Second, we need to check that $\int_{-\infty}^{\infty} f(x)\, fx = 1$:

$$\int_{-\infty}^{\infty} f(x)\, fx = \int_{-\infty}^{1} 0\, dx + \int_{1}^{\infty} 2x^{-3}\, dx = 0 + 2 \cdot \left. \frac{x^{-3+1}}{-3+1} \right|_1^{\infty} = 2 \cdot \left. \frac{-1}{2} x^{-2} \right|_1^{\infty}$$
$$= -\left. \frac{1}{x^2} \right|_1^{\infty} = \left[-\frac{1}{\infty^2} \right] - \left[-\frac{1}{1^2} \right] = [-0] - [-1] = 1$$

Hence $f(x)$ meets both requirements to be a valid density function. ■

2. Compute the expected value, $E(X)$, and variance, $V(X)$, of X. *[6]*

SOLUTION. We compute $E(X)$ first, since we'll need it to compute $V(X)$:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{1} x \cdot 0\, dx + \int_{1}^{\infty} 2x^{-3} \cdot x\, dx = \int_{-\infty}^{1} 0\, dx + \int_{1}^{\infty} 2x^{-2}\, dx$$
$$= 0 + 2 \cdot \left. \frac{x^{-2+1}}{-2+1} \right|_1^{\infty} = -2x^{-1}\Big|_1^{\infty} = -\left. \frac{2}{x} \right|_1^{\infty} = \left[-\frac{2}{\infty} \right] - \left[-\frac{2}{1} \right] = [-0] - [-2] = 2$$

Since $V(X) = E\left(X^2\right) - [E(X)]^2$, we need to compute $E\left(X^2\right)$:

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{-\infty}^{1} x^2 \cdot 0\, dx + \int_{1}^{\infty} 2x^{-3} \cdot x^2\, dx = \int_{-\infty}^{1} 0\, dx + \int_{1}^{\infty} 2x^{-1}\, dx$$
$$= 0 + 2 \int_{1}^{\infty} \frac{1}{x}\, dx = 2\ln(x)\Big|_1^{\infty} = 2\ln(\infty) - 2\ln(1) = 2 \cdot \infty - 2 \cdot 0 = \infty - 0 = \infty$$

It follows that $E\left(X^2\right)$, and hence also $V(X)$, is undefined. ■