Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 2018

Solution to Assignment # 6
A Continuous Density?

1. Determine whether $f(x) = \begin{cases} \frac{1}{2}e^{x} & x \le 0 \\ \frac{1}{2}e^{-x} & x \ge 0 \end{cases}$ is a valid probability density. *[10]*

SOLUTION. First, since $e^t > 0$ for all real numbers t, $f(x) > 0$ (and thus also $f(x) \ge 0$) for all x by the definition of $f(x)$. (Each part of the definition of $f(x)$ is $\frac{1}{2}e^t$, where $t = \pm x$.) Thus $f(x)$ satisfies the first part of the definition of a valid density function.

Second, in somewhat excessive detail,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} \frac{1}{2}e^{x}\,dx + \int_{0}^{\infty} \frac{1}{2}e^{-x}\,dx = \frac{1}{2}\int_{-\infty}^{0} e^{x}\,dx + \frac{1}{2}\int_{0}^{\infty} e^{-x}\,dx$$

The antiderivative of e^x is just e^x; for the second integral we use the substitution $u = -x$, so $du = -dx$, and thus $dx = (-1)\,du$. We will keeep the limits in terms of x and substitute back after finding the antiderivative.

$$\begin{aligned} &= \frac{1}{2}e^{x}\Big|_{-\infty}^{0} + \frac{1}{2}\int_{x=0}^{x=\infty} e^{u}(-1)\,du = \left[\frac{1}{2}e^{0} - \frac{1}{2}e^{-\infty}\right] + (-1)\frac{1}{2}\int_{x=0}^{x=\infty} e^{u}\,du \\ &= \left[\frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 0\right] - \frac{1}{2}e^{u}\Big|_{x=0}^{x=\infty} = \frac{1}{2} - \frac{1}{2}e^{-x}\Big|_{0}^{\infty} = \frac{1}{2} - \left[\frac{1}{2}e^{-\infty} - \frac{1}{2}e^{0}\right] \\ &= \frac{1}{2} - \left[\frac{1}{2}\cdot 0 - \frac{1}{2}\cdot 1\right] = \frac{1}{2} - \left[-\frac{1}{2}\right] = \frac{1}{2} + \frac{1}{2} = 1\,, \end{aligned}$$

as is also needed for a valid density function.

Since it satisfies both requirements, $f(x)$ is a valid density function. ■