Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 2018
Solutions to Assignment #5
Simulating coins.

1. You are stuck on a desert island with a friend and single coin which is your only source of randomness. [Deterministic desert islands are very boring!] The coin in question is biased in that it comes up heads more often than tails, though you do not know the exact bias. (It does come up tails some of the time.) You and your friend want to play a game that needs a fair coin. How can you use the biased coin to simulate the fair coin that you need? *[5]*

SOLUTION. The coin with unknown bias can be used to simulate a fair coin as follows:

1. Toss the biased coin twice.
2. If the outcome of the double toss is HT, return H for the toss of the fair coin, if the outcome of the double toss is TH, return T for the toss of the fair coin, and if the outcome of the double toss is HH or TT, go back to step one and repeat the experiment.

Why does this work? First, whatever the bias of the given coin may be, the two tosses of this coin are independent of each other, so $P(HT) = P(H) \cdot P(T) = P(T) \cdot P(H) = P(TH)$. This means that the experiment has an equal probability of returning a H as it does returning a T for the simulated fair coin, as desired. Second, the probability that the experiment will have to repeat at least n times is $[P(HH) + P(TT)]^n$, and because $P(HH) + P(TT) < 1$ for the biased coin [Why?], it follows that the probability it will repeat forever is 0. Thus the given process will generate an outcome for the fair coin being simulated eventually. □

2. Having been rescued, you and your friend get stuck on another desert island with a single coin, but this time it's a coin that is known to be fair. Having gotten bored of the game needing a fair coin on the last desert island adventure, you and your friend would like to play a game that requires a biased coin, in particular, a coin with $P(H) = 0.6$ and $P(T) = 0.4$. How can you use the fair coin to simulate the biased coin you need? *[5]*

SOLUTION. The fair coin can be used to simulate the desired biased coin as follows:

1. Toss the fair coin three times.
2. If the outcome of the three tosses is HHH, HHT, or HTH, return H for the toss of the biased coin, if the outcome of the three tosses is THH or HTT, return T for the toss of the biased coin, and if the outcome of the three tosses is THT, TTH, or TTT, go back to step one and repeat the experiment.

Why does this work? Since every sequence of three tosses of the fair coin has an equal probability, the probability of getting a H for a toss of the simulated biased coin is $\frac{P(HHH)+P(HHT)+P(HTH)}{P(HHH)+P(HHT)+P(HTH)+P(THH)+P(HTT)} = \frac{3/8}{5/8} = \frac{3}{5}$, and the probability of getting a T for the toss of teh simulated biased coin is $\frac{P(THH)+P(HTT)}{P(HHH)+P(HHT)+P(HTH)+P(THH)+P(HTT)} = \frac{2/8}{5/8} = \frac{2}{5}$, as desired. The process will eventually generate an outcome for the simulated coin in the same way the process given in the solution to **1** above will. ■