Mathematics 1550H – Introduction to probability

Trent University, Winter 2018

Solutions to the Final Examination

Tuesday, 17 April, 2018

Spatio-temporal locus: 19:00–22:00 in the Gym *Inflicted by* Стефан Біланюк.

Instructions: Do both of parts **Bernoulli** and **Chebyshev**, and, if you wish, part **Dopey**. Show all your work and simplify answers as much as practical. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; standard normal table; lots of neurons.

Part Bernoulli. Do all of **1**–**5**. *[Subtotal = 68/100]*

1. A fair six-sided non-standard die has faces numbered 0, 1, 1, 2, 2, and 2, respectively. The random variable X records the number that comes up on a single roll of the die.

a. What is the probability function of X? *[5]*

b. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[5]*

Solutions. **a.** Since the die is fair, each of the six faces has an equal probability of $\frac{1}{6}$ of coming up. If X returns the number on the face that comes up, then the probability function of X is:

$$m(k) = P(X = k) = \frac{\#\text{ faces with } k}{6} = \begin{cases} \frac{1}{6} & k = 0 \\ \frac{2}{6} & k = 1 \\ \frac{3}{6} & k = 2 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{6} & k = 0 \\ \frac{1}{3} & k = 1 \\ \frac{1}{2} & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

For those who like formulas, this comes down to $m(k) = \begin{cases} \frac{1}{6}(k+1) & k = 0,\, 1,\, 2 \\ 0 & \text{otherwise} \end{cases}$. □

b. By definition:

$$E(X) = \sum_{k=0}^{2} kP(X = k) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{2} = 0 + \frac{1}{3} + 1 = \frac{4}{3}$$

$$E\left(X^2\right) = \sum_{k=0}^{2} k^2 P(X = k) = 0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{2} = 0 + \frac{1}{3} + 2 = \frac{7}{3}$$

$$V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{7}{3} - \left[\frac{4}{3}\right]^2 = \frac{21}{9} - \frac{16}{9} = \frac{5}{9} \quad \blacksquare$$

2. Let T be a continuous random variable with the following probability density function:

$$f(t) = \begin{cases} |t| & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $f(t)$ is indeed a probability density function. *[8]*

b. If you know that $T \leq \frac{1}{2}$, what is the probability that $T \geq 0$? *[7]*

SOLUTIONS WITH CALCULUS. **a.** First, $f(t) = |t| \geq 0$ when $-1 \leq t \leq 1$, and $f(t) = 0 \geq 0$ otherwise, so $f(t) \geq 0$ for all t.

Second, we need to check that $\int_{-\infty}^{\infty} f(t)\,dt = 1$. Note that $|t| = \begin{cases} t & t \geq 0 \\ -t & t \leq 0 \end{cases}$, which allows us to get rid of the absolute value by breaking up the integral appropriately.

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\,dt &= \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{1} |t|\,dt + \int_{1}^{\infty} 0\,dt = 0 + \int_{-1}^{0} (-t)\,dt + \int_{0}^{1} t\,dt + 0 \\ &= -\frac{t^2}{2}\Big|_{-1}^{0} + \frac{t^2}{2}\Big|_{0}^{1} = \left(-\frac{0^2}{2}\right) - \left(-\frac{(-1)^2}{2}\right) + \frac{1^2}{2} - \frac{0^2}{2} \\ &= 0 - \left(-\frac{1}{2}\right) + \frac{1}{2} - 0 = \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

Thus $f(t)$ satisfies both of the defining conditions to be a probability density function. □

b. This is a conditional probability problem: it is asking for the probability of $T \geq 0$ given $T \leq \frac{1}{2}$. Since $P\left(T \geq 0 \mid T \leq \frac{1}{2}\right) = \frac{P\left(T \geq 0 \,\&\, T \leq \frac{1}{2}\right)}{P\left(T \leq \frac{1}{2}\right)}$, we need to compute both $P\left(T \leq \frac{1}{2}\right)$ and $P\left(T \geq 0 \,\&\, T \leq \frac{1}{2}\right)$.

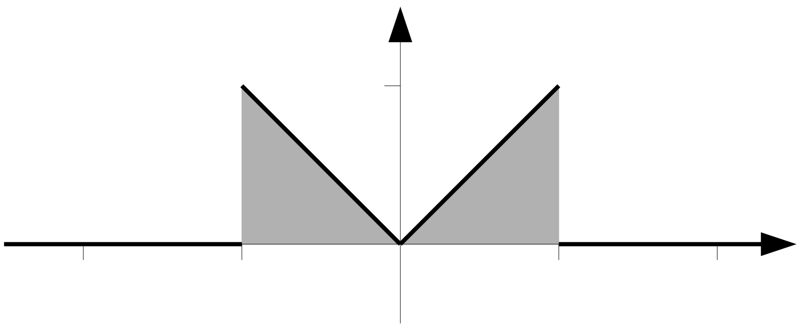
$$\begin{aligned} P\left(T \leq \frac{1}{2}\right) &= \int_{-\infty}^{1/2} f(t)\,dt = \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{1/2} |t|\,dt = 0 + \int_{-1}^{0} (-t)\,dt + \int_{0}^{1/2} t\,dt \\ &= -\frac{t^2}{2}\Big|_{-1}^{0} + \frac{t^2}{2}\Big|_{0}^{1/2} = \left(-\frac{0^2}{2}\right) - \left(-\frac{(-1)^2}{2}\right) + \frac{\left(\frac{1}{2}\right)^2}{2} - \frac{0^2}{2} \\ &= 0 - \left(-\frac{1}{2}\right) + \frac{\frac{1}{4}}{2} - 0 = \frac{1}{2} + \frac{1}{4} \div 2 = \frac{1}{2} + \frac{1}{8} = \frac{5}{8} \end{aligned}$$

$$\begin{aligned} P\left(T \geq 0 \,\&\, T \leq \frac{1}{2}\right) &= P\left(0 \leq T \leq \frac{1}{2}\right) = \int_{0}^{1/2} f(t)\,dt = \int_{0}^{1/2} |t|\,dt = \int_{0}^{1/2} t\,dt \\ &= \frac{t^2}{2}\Big|_{0}^{1/2} = \frac{\left(\frac{1}{2}\right)^2}{2} - \frac{0^2}{2} = \frac{\frac{1}{4}}{2} - 0 = \frac{1}{4} \div 2 = \frac{1}{8} \end{aligned}$$

It follows that $P\left(T \geq 0 \mid T \leq \frac{1}{2}\right) = \frac{P\left(T \geq 0 \,\&\, T \leq \frac{1}{2}\right)}{P\left(T \leq \frac{1}{2}\right)} = \frac{1/8}{5/8} = \frac{1}{8} \cdot \frac{8}{5} = \frac{1}{5}$. ■

SOLUTIONS WITHOUT CALCULUS. **a.** First, $f(t) = |t| \geq 0$ when $-1 \leq t \leq 1$, and $f(t) = 0 \geq 0$ otherwise, so $f(t) \geq 0$ for all t.

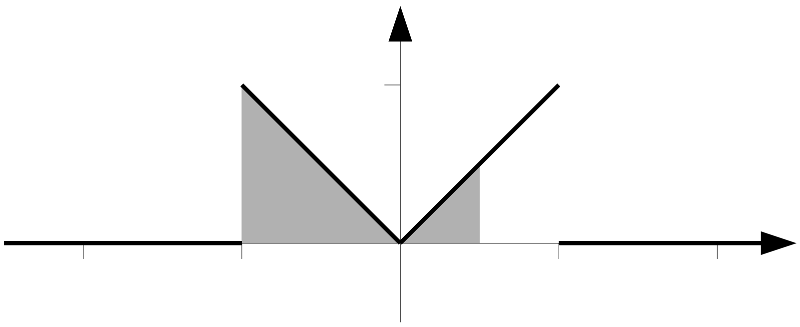
Second, we need to check that $\int_{-\infty}^{\infty} f(t)\,dt = 1$. This integral is just the area of the region under the graph of $y = f(t)$ and above the horizontal axis. Consider the graph of $y = f(t) = \begin{cases} |t| & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$:



Outside of $-1 \leq t \leq 1$ the graph lies right on the horizontal axis and contributes no area. For $-1 \leq t \leq 1$ the area under the graph consists of two right triangles, each with base 1 and height 1. The area under the graph is therefore $\frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} + \frac{1}{2} = 1$, as required.

Thus $f(t)$ satisfies the defining conditions to be a probability density function. $\square$

b. This is a conditional probability problem: it is asking for the probability of $T \geq 0$ given $T \leq \frac{1}{2}$. Since $P\left(T \geq 0 \mid T \leq \frac{1}{2}\right) = \frac{P\left(T \geq 0 \ \&\ T \leq \frac{1}{2}\right)}{P\left(T \leq \frac{1}{2}\right)}$, we need to compute both $P\left(T \leq \frac{1}{2}\right)$ and $P\left(T \geq 0 \ \&\ T \leq \frac{1}{2}\right)$. Again, consider the graph of $y = f(t)$.



$P\left(T \geq 0 \ \&\ T \leq \frac{1}{2}\right)$ is the area under the graph for $0 \leq t \leq \frac{1}{2}$, which is the area of a right triangle with base $\frac{1}{2}$ and height $\frac{1}{2}$. It follows that $P\left(T \geq 0 \ \&\ T \leq \frac{1}{2}\right) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

$P\left(T \leq \frac{1}{2}\right)$ is the area under the graph for $-\infty < t \leq \frac{1}{2}$. For $-\infty < t < -1$ the graph lies right on the horizontal axis and contributes no area, for $-1 \leq t \leq 0$ the area under the graph is the area of a right triangle with base 1 and height 1, and for $0 \leq t \leq \frac{1}{2}$ the area under the graph is the area of a right triangle with base $\frac{1}{2}$ and height $\frac{1}{2}$. It follows that $P\left(T \leq \frac{1}{2}\right) = 0 + \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$.

Thus $P\left(T \geq 0 \mid T \leq \frac{1}{2}\right) = \frac{P\left(T \geq 0 \ \&\ T \leq \frac{1}{2}\right)}{P\left(T \leq \frac{1}{2}\right)} = \frac{1/8}{5/8} = \frac{1}{8} \cdot \frac{8}{5} = \frac{1}{5}$. $\blacksquare$

3. A fair coin is tossed until it comes up heads, then tossed some more until it comes up tails. The random variable Y counts the total number of times the coin is tossed during the experiment.

a. Find the probability function, expected value, and variance of Y. *[12]*

b. Use Chebyshev's Inequality to estimate the probability that $Y \geq 9$. *[5]*

c. Compute the probability that $Y \geq 9$. *[5]*

SOLUTIONS. **a.** Although the definition of "success" changes after the first success, the probability of success does not because the coin is fair so $P(H) = P(T) = \frac{1}{2}$. The experiment therefore amounts to repeating a Bernoulli trial with probability of success $p = \frac{1}{2}$ until the second success. If Y counts the number of tosses required to achieve the second success, it follows that Y has a negative binomial dstribution with $p = \frac{1}{2}$, $q = 1 - \frac{1}{2} = \frac{1}{2}$, and $k = 2$. Thus the probability function of Y is

$$m(y) = \binom{y-1}{k-1} p^k q^{y-k} = \binom{y-1}{2-1} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{y-2} = \binom{y-1}{1} \left(\frac{1}{2}\right)^y = \frac{y-1}{2^y},$$

and the expected value and variance of Y are

$$E(Y) = \frac{k}{p} = \frac{2}{1/2} = 2 \cdot \frac{2}{1} = 4 \quad \text{and} \quad V(Y) = \frac{kq}{p^2} = \frac{2 \cdot (1/2)}{(1/2)^2} = \frac{1}{1/4} = 1 \cdot \frac{4}{1} = 4. \quad \square$$

b. Recall that Chebyshev's Inequality states that for a random variable X with $E(X) = \mu$ and $V(X) = \sigma^2$ and any $\varepsilon > 0$, we have $P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}$. From part **a**, we have $E(Y) = 4$ and $V(Y) = 4$, so it follows that

$$P(Y \geq 9) = P(Y - 4 \geq 5) \leq P(|Y - 4| \geq 5) \leq \frac{4}{5^2} = \frac{4}{25} = 0.16. \quad \square$$

c. Here we go:

$$\begin{aligned}
P(Y \geq 9) &= 1 - P(Y < 9) = 1 - [P(Y=2) + P(Y=3) + P(Y=4) + P(Y=5) \\
&\qquad\qquad\qquad\qquad + P(Y=6) + P(Y=7) + P(Y=8)] \\
&= 1 - [m(2) + m(3) + m(4) + m(5) + m(6) + m(7) + m(8)] \\
&= 1 - \left[\frac{2-1}{2^2} + \frac{3-1}{2^3} + \frac{4-1}{2^4} + \frac{5-1}{2^5} + \frac{6-1}{2^5} + \frac{7-1}{2^7} + \frac{8-1}{2^8}\right] \\
&= 1 - \left[\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + \frac{6}{128} + \frac{7}{256}\right] \\
&= 1 - \left[\frac{64}{256} + \frac{64}{256} + \frac{48}{256} + \frac{32}{256} + \frac{20}{256} + \frac{12}{256} + \frac{7}{256}\right] \\
&= 1 - \frac{247}{256} = \frac{9}{256} \quad \blacksquare
\end{aligned}$$

NOTE: As a small sanity check, observe that $\frac{9}{256} = 0.03515625 < 0.16 = \frac{4}{25}$.

4. A hand of five cards is randomly chosen, simultaneously and without replacement, from a standard 52-card deck.

a. What is the probability that the hand includes all four of one kind*? *[5]*

b. What is the probability that the hand includes at least three of one kind? *[5]*

c. What is the probability that the hand includes four of one kind, given that it includes at least three of one kind? *[5]*

SOLUTIONS. Note that since the hand is being chosen simultaneously and without replacement, order doesn't matter, so there are $\binom{52}{5} = 2598000$ equally likely possible hands.

a. There are $\binom{13}{1} = 13$ ways to choose the kind for the four of a kind, $\binom{4}{4} = 1$ way to choose the four of that kind, and $\binom{52-4}{1} = \binom{48}{1} = 48$ ways to choose a card of another kind for the fifth card of the hand. It follows that there are $\binom{13}{1}\binom{4}{4}\binom{48}{1} = 13 \cdot 1 \cdot 48 = 624$ hands which include all four of one kind, and thus the probability of a randomly chosen hand including four of one kind is $P(4 \text{ of a kind}) = \frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} = \frac{624}{2598000} \approx 0.00024$. $\square$

b. We first compute the probability of getting exactly three of one kind. There are $\binom{13}{1} = 13$ ways to choose the kind for the three of a kind, $\binom{4}{3} = 4$ ways to choose three of that kind, and $\binom{52-4}{2} = \binom{48}{2} = 1128$ ways to choose two more cards of another kind or kinds for the rest of the hand. It follows that there are $\binom{13}{1}\binom{4}{3}\binom{48}{2} = 13 \cdot 4 \cdot 1128 = 58656$ hands which include exactly three of one kind, and thus the probability of a randomly chosen hand including exactly of one kind is $P(3 \text{ of a kind}) = \frac{\binom{13}{1}\binom{4}{4}\binom{48}{2}}{\binom{52}{5}} = \frac{58656}{2598000} \approx 0.02258$.

The probability of getting at least three of one kind is the sum of the probabilities of getting exactly three of one kind and of getting four of one kind:

$$\begin{aligned} P(\geq 3 \text{ of a kind}) = P(3 \text{ of a kind}) + P(4 \text{ of a kind}) &= \frac{\binom{13}{1}\binom{4}{4}\binom{48}{2}}{\binom{52}{5}} + \frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \\ &= \frac{58656}{2598000} + \frac{624}{2598000} = \frac{59280}{2598000} \approx 0.02282 \quad \square \end{aligned}$$

c. This is a conditional probability problem.

$$\begin{aligned} P(4 \text{ of a kind} \mid \geq 3 \text{ of a kind}) &= \frac{P(4 \text{ of a kind and} \geq 3 \text{ of a kind})}{P(\geq 3 \text{ of a kind})} \\ &= \frac{P(4 \text{ of a kind})}{P(\geq 3 \text{ of a kind})} = \frac{\binom{13}{1}\binom{4}{4}\binom{48}{2} \div \binom{52}{5}}{\left[\binom{13}{1}\binom{4}{4}\binom{48}{2} + \binom{13}{1}\binom{4}{4}\binom{48}{1}\right] \div \binom{52}{5}} \\ &= \frac{\binom{13}{1}\binom{4}{4}\binom{48}{2}}{\binom{13}{1}\binom{4}{4}\binom{48}{2} + \binom{13}{1}\binom{4}{4}\binom{48}{1}} = \frac{624}{59280} \approx 0.01053 \quad \blacksquare \end{aligned}$$

* Recall that the *kinds* are A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, and 2. The *suits* are ♡, ◇, ♣, and ♠.

5. Suppose U is a continuous random variable that has a normal distribution with expected value $\mu = -2$ and variance $\sigma^2 = 1$. Compute $P(-1.1 \leq U \leq 1.1)$ with the help of a standard normal table. *[6]*

SOLUTION. Recall that if U has a normal distribution with expected value $\mu = -2$ and variance $\sigma^2 = 1$, then $Z = \frac{U-\mu}{\sigma} = \frac{U-(-2)}{1} = U + 2$ has a standard normal distribution. Thus:

$$\begin{aligned} P(-1.1 \leq U \leq 1.1) &= P\left(\frac{-1.1-(-2)}{1} \leq \frac{U-(-2)}{1} \leq \frac{1.1-(-2)}{1}\right) \\ &= P(-1.1+2 \leq U+2 \leq 1.1+2) \\ &= P(0.9 \leq Z \leq 3.1) \\ &= P(Z \leq 3.1) - P(Z < 0.9) \\ &\qquad \text{... which we look up in the standard normal table:} \\ &\approx 0.9990 - 0.8159 = 0.1831 \quad \blacksquare \end{aligned}$$

*[Parts **Chebyshev** and **Dopey** are on page 2.]*

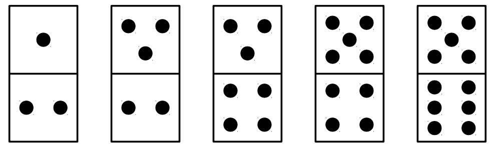
Part Chebyshev. Do any *two* (2) of **6–9**. *[Subtotal = 32/100]*

6. You are given five dominoes, marked as in the figure at right.

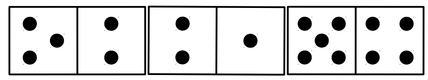
a. How many ways are there to choose three of the dominoes and lay them out end-to-end? *[8]*

b. If three of the dominoes are chosen at random and laid out end-to-end randomly, what is the probability that both pairs of adjacent ends will match? *[8]*

The five dominoes:



Three dominoes end-to-end, with one pair of adjacent ends matching:



SOLUTIONS. **a.** There are $\binom{5}{3} = 10$ ways to choose three out five dominoes and $3\dot{2} \cdot 1 = 3! = 6$ ways to put them in order. Each domino can also be laid out in two ways by swapping which end faces which ways (for example, the first domino can be laid out as $[1, 2]$ or as $[2, 1]$), meaning that there $2^3 = 8$ ways to decide which way to orient three ordered dominoes. It follows that there are $\binom{5}{3} \cdot 3! \cdot 2^3 = 10 \cdot 6 \cdot 8 = 480$ ways to choose three of the five dominoes and arrange them end-to-end. □

b. There are only six ways three of the five dominoes can be laid out so that both pairs of adjacent ends match: $[1, 2][2, 3][3, 4]$, $[2, 3][3, 4][4, 5]$, and $[3, 4][4, 5][5, 6]$, and their reverses $[4, 3][3, 2][2, 1]$, $[5, 4][4, 3][3, 2]$, and $[6, 5][5, 4][4, 3]$. Since, according to the solution for part **a**, there are 480 possible end-to-end arrangements of three of the five dominoes, the probability that a random arrangement will have both pairs of adjacent ends match is $\frac{6}{480} = \frac{1}{80} = 0.0125$. ■

7. A frog and a toad have a race of sorts from one end of a straight 3 m track to the other. The frog hops 10 cm at a time and the toad hops 15 cm at a time, and they both head straight from the starting line to the finish line. However, the frog and toad do not move all the time. Every 2 seconds the frog, with equal probability, either makes a hop toward the finish line or stays still; every 3 seconds the toad, with equal probability, either makes a hop toward the finish line or stays still. Assuming they begin at the starting line at the same time, which would you expect to reach the finish line first? *[16]*

SOLUTION. The track is $3\ m = 300\ cm$ long, so the frog will need to complete $\frac{300}{10} = 30$ jumps to complete the race, while the toad will need to complete $\frac{300}{15} = 20$ jumps to complete the race. For both the toad and the frog, at each respective time interval, there is a probability of $\frac{1}{2}$ that a jump will be taken. In each case, this gives a negative binomial distribution with the probability of success being $p = \frac{1}{2}$, with $k = 30$ required successes for the frog and $k = 20$ required successes for the toad, for the number of attempts required for each to reach the finish line. For the frog this means that the expected number of attempts required is $\frac{k}{p} = \frac{30}{1/2} = 60$, and for the toad this means that the expected number of atempts required is $\frac{k}{p} = \frac{20}{1/2} = 40$. Since the frog always takes 2 s between attempts to jump, the frog would be expected to cross the finish line after $60 \cdot 2 = 120\ s$; similarly, since the toad always takes 3 s between attempts to jump, the frog would be expected to cross the finish line after $40 \cdot 3 = 120\ s$. Thus neither the frog nor the toad can expect to reach the finish line first ... ■

8. Suppose X_1 and X_2 are independent continuous random variables that each have an exponential distribution with $\lambda = 1$. Let $X = X_1 + X_2$.

a. Compute the expected value, $E(X)$, and variance, $V(X)$, of X. *[6]*

b. Find the probability density function of X. *[10]*

SOLUTIONS. **a.** Since X_1 and X_2 have exponential distributions with $\lambda = 1$, we have $E(X_1) = E(X_2) = \frac{1}{\lambda} = \frac{1}{1} = 1$ and $V(X_1) = V(X_2) = \frac{1}{\lambda^2} = \frac{1}{1^2} = 1$. It follows that $E(X) = E(X_1) + E(X_2) = 1 + 1 = 2$ and, since X_1 and X_2 are also independent, that $V(X) = V(X_1) + V(X_2) = 1 + 1 = 2$. $\square$

b. Since X_1 and X_2 have exponential distributions with $\lambda = 1$, both use the probability density function $f(x) = \begin{cases} \lambda e^{-\lambda x} & t \geq 0 \\ 0 & t < 0 \end{cases} = \begin{cases} e^{-x} & t \geq 0 \\ 0 & t < 0 \end{cases}$. The probability density function of $X = X_1 + X_2$ is therefore the convolution of f with itself:

$$
\begin{aligned}
(f * f)(x) &= \int_{-\infty}^{\infty} f(x-t)f(t)\,dt = \int_{-\infty}^{0} f(x-t) \cdot 0\,dt + \int_{0}^{\infty} f(x-t)e^{-t}\,dt \\
&= \int_{-\infty}^{0} 0\,dt + \begin{cases} \int_0^x e^{-(x-t)}e^{-t}\,dt + \int_x^\infty 0 \cdot e^{-t}\,dt & x \geq 0 \\ \int_0^\infty 0 \cdot e^{-t}\,dt & x < 0 \end{cases} \\
&= 0 + \begin{cases} \int_0^x e^{-x+t-t}\,dt + \int_x^\infty 0dt & x \geq 0 \\ \int_0^\infty 0\,dt & x < 0 \end{cases} \\
&= \begin{cases} \int_0^x e^{-x}\,dt + 0 & x \geq 0 \\ 0 & x < 0 \end{cases} \quad \text{but } x \text{ is a constant to } t \text{, so} \\
&= \begin{cases} e^{-x} \int_0^x 1\,dt & x \geq 0 \\ 0 & x < 0 \end{cases} = \begin{cases} e^{-x}\, t\big|_0^x 1\,dt & x \geq 0 \\ 0 & x < 0 \end{cases} \\
&= \begin{cases} e^{-x}(x-0) & x \geq 0 \\ 0 & x < 0 \end{cases} = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \blacksquare
\end{aligned}
$$

9. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$Y \backslash X$	2	3	4
-1	0.1	0.2	0.2
0	0.2	0	0
1	0.2	0	0.1

a. Compute the expected values $E(X)$ and $E(Y)$, the variances $V(X)$ and $V(Y)$, and also the covariance $\text{Cov}(X, Y)$ of X and Y. *[10]*

b. Determine whether X and Y are independent. *[2]*

c. Let $W = 2X + Y$. Compute $E(W)$ and $V(W)$. *[4]*

SOLUTIONS. **a.** We're off to do arithmetic, arithmetic ...

$$\begin{aligned}
E(X) &= \sum_x xP(X = x) = 2(0.1 + 0.2 + 0.2) + 3(0.2 + 0 + 0) + 4(0.2 + 0 + 0.1) \\
&= 2 \cdot 0.5 + 3 \cdot 0.2 + 4 \cdot 0.3 = 1 + 0.6 + 1.2 = 2.8 \\
E(Y) &= \sum_y yP(Y = y) = (-1)(0.1 + 0.2 + 0.2) + 0(0.2 + 0 + 0) + 1(0.2 + 0 + 0.1) \\
&= -1 \cdot 0.5 + 0 \cdot 0.2 + 1 \cdot 0.3 = -0.5 + 0 + 0.3 = -0.2 \\
E\left(X^2\right) &= \sum_x x^2 P(X = x) = 2^2(0.1 + 0.2 + 0.2) + 3^2(0.2 + 0 + 0) + 4^2(0.2 + 0 + 0.1) \\
&= 4 \cdot 0.5 + 9 \cdot 0.2 + 16 \cdot 0.3 = 2 + 1.8 + 4.8 = 8.6 \\
E\left(Y^2\right) &= \sum_y y^2 P(Y = y) = (-1)^2(0.1 + 0.2 + 0.2) + 0^2(0.2 + 0 + 0) + 1^2(0.2 + 0 + 0.1) \\
&= 1 \cdot 0.5 + 0 \cdot 0.2 + 1 \cdot 0.3 = 0.5 + 0 + 0.3 = 0.8 \\
E(XY) &= \sum_{x,y} xyP(X = x \ \& \ Y = y) \\
&= (-1) \cdot 2 \cdot 0.1 + (-1) \cdot 3 \cdot 0.2 + (-1) \cdot 4 \cdot 0.2 \\
&\quad + 0 \cdot 2 \cdot 0.2 + 0 \cdot 3 \cdot 0 + 0 \cdot 4 \cdot 0 \\
&\quad + 1 \cdot 2 \cdot 0.2 + 1 \cdot 3 \cdot 0 + 1 \cdot 4 \cdot 0.1 \\
&= [(-0.2) + (-0.6) + (-0.8)] + [0 + 0 + 0] + [0.4 + 0 + 0.4] \\
&= -1.6 + 0 + 0.8 = -0.8 \\
V(X) &= E\left(X^2\right) - [E(X)]^2 = 8.6 - (2.8)^2 = 8.6 - 7.84 = 0.76 \\
V(Y) &= E\left(Y^2\right) - [E(Y)]^2 = 0.8 - (-0.2)^2 = 0.8 - 0.04 = 0.76 \\
\text{Cov}(X, Y) &= E(XY) - E(X) \cdot E(Y) = (-0.8) - 2.8 \cdot (-0.2) = -0.8 + 0.56 = -0.24 \quad \square
\end{aligned}$$

b. Since $\text{Cov}(X, Y) = -0.24 \neq 0$, X and Y cannot be independent. $\square$

c. $E(W) = E(2X + Y) = 2E(X) + E(Y) = 2 \cdot 2.8 + (-0.2) = 5.6 - 0.2 = 5.4$ and

$$\begin{aligned}
V(W) &= V(2X + Y) = V(2X) + V(Y) + 2 \cdot \text{Cov}(2X, Y) \\
&= 2^2 \cdot V(X) + V(Y) + 2 \cdot 2 \cdot \text{Cov}(X, Y) \\
&= 4 \cdot 0.76 + 0.76 + 4 \cdot (-0.24) = 3.04 + 0.76 - 0.96 = 2.84 \quad \square
\end{aligned}$$

[Total = 100]

Part Dopey. Bonus!

⚀. Two fair standard dice are rolled simultaneously three times. What is the probability that they will come up with the same face on at least one of the three rolls? *[1]*

⚅. Write a haiku touching on probability or mathematics in general. *[1]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

[Part **Bernoulli** *is on page 1.]*

I HOPE THAT YOU ENJOYED THE COURSE. HAVE A GOOD SUMMER!