Trent University, Winter 2017

MATH 1550H Test

Thursday, 2 March, 2017

Time: 50 minutes

Name: ______________________________

Student Number: ______________________________

Question	Mark	
1	____	
2	____	
3	____	
Total	____	/30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Suppose the continuous random variable X has $f(x) = \begin{cases} 1+x & -1 \leq x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ as its probability density function. Compute $P(-0.5 \leq X \leq 0.5)$.

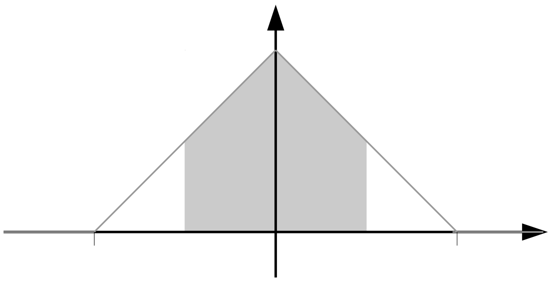
b. A hand of five cards is drawn simultaneously and randomly from a standard 52-card deck. What is the probability that the hand includes exactly three $\heartsuit$s?

c. A fair coin is tossed four times. What is the probability that at least two heads will come up?

Solutions. **a.** *(With calculus.)* By definition:

$$\begin{aligned} P(-0.5 \leq X \leq 0.5) &= \int_{-0.5}^{0.5} f(x)\,dx = \int_{-0.5}^{0} (1+x)\,dx + \int_{0}^{0.5} (1-x)\,dx \\ &= \left(x + \frac{x^2}{2}\right)\Big|_{-0.5}^{0} + \left(x - \frac{x^2}{2}\right)\Big|_{0}^{0.5} \\ &= \left[\left(0 + \frac{0^2}{2}\right) - \left(-0.5 + \frac{(-0.5)^2}{2}\right)\right] + \left[\left(0.5 - \frac{0.5^2}{2}\right) - \left(0 - \frac{0^2}{2}\right)\right] \\ &= [0 - (-0.375)] + [0.375 - 0] = 0.75 = \frac{3}{4} \qquad \Box \end{aligned}$$

a. *(Without calculus.)* By definition, $P(-0.5 \leq X \leq 0.5)$ is the area under the graph of the density function of X between -0.5 and 0.5. A sketch of this region



shows that it consists of a rectangle of width $1 = 0.5-(-0.5)$ and height $0.5 = 0.5-0$, and hence area $1 \cdot 0.5 = 0.5$, under a triangle with base $1 = 0.5-(-0.5)$ and height $0.5 = 1-0.5$, and hence area $\frac{1}{2} \cdot 1 \cdot 0.5 = 0.25$. The area of this region is thus $0.5 + 0.25 = 0.75$, and therefore $P(-0.5 \leq X \leq 0.5) = 0.75$. $\Box$

b. There are $\binom{13}{3}$ ways to choose three $\heartsuit$s from the 13 that are in the hand, and $\binom{39}{2}$ ways to choose two cards from the remaining 39 cards in the deck. Since there are a total of $\binom{52}{5}$ ways to choose a hand of 5 cards from a standard 52-card deck, and each hand is equally likely to be chosen, the probability that a randomly and simultaneously chosen hand includes exactly 3 $\heartsuit$s is $\dfrac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}}$. $\Box$

c. Note that $P(\geq 2\ H) = 1 - P(< 2\ H) = 1 - [P(0\ H) + P(1\ H)]$. Since $P(0\ H) = P(TTTT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$ and $P(1\ H) = P(HTTT) + P(THTT) + P(TTHT) + P(TTTH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \cdots + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16} + \cdots + \frac{1}{16} = \frac{4}{16}$, it follows that $P(\geq 2\ H) = 1 - [P(0\ H) + P(1\ H)] = 1 - \left[\frac{1}{16} + \frac{4}{16}\right] = 1 - \frac{5}{16} = \frac{11}{16}$. ■

2. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. Show that if A and B are events in some sample space, with $P(A) > 0$ and $P(B) > 0$, then $\dfrac{P(A|B)}{P(A)} = \dfrac{P(B|A)}{P(B)}$.

b. Determine whether $g(x) = \begin{cases} x^{-2} & 1 \leq x \\ 0 & x < 1 \end{cases}$ is a valid continuous probability density.

c. A fair non-standard six-sided die thas one face numbered 1, two faces numbered 2, and three faces numbered 3. What is the expected value of the number that comes up if the die is rolled once?

SOLUTIONS. **a.** By definition, $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, so $P(A|B) = P(A \cap B)P(B)$, and thus $P(A \cap B) = \dfrac{P(A|B)}{P(B)}$. Similarly, $P(B|A) = \dfrac{P(B \cap A)}{P(A)}$, so $P(B|A) = P(B \cap A)P(A)$, and thus $P(B \cap A) = \frac{P(B|A)}{P(A)}$, and thus $P(B \cap A) = \dfrac{P(B|A)}{P(A)}$. Since $A \cap B = B \cap A$, it follows that $\dfrac{P(A|B)}{P(A)} = P(A \cap B) = P(B \cap A) = \dfrac{P(B|A)}{P(B)}$. $\square$

b. First, when $x < 1$, $g(x) = 0 \geq 0$, and when $x \geq 1$, $g(x) = x^{-2} = \frac{1}{x^2} > 0 \geq 0$, so $g(x) \geq 0$ for all x. Second,

$$\begin{aligned} \int_{-\infty}^{\infty} g(x)\,dx &= \int_{-\infty}^{1} 0\,dx + \int_{1}^{\infty} x^{-2}\,dx = 0 + \left.\frac{x^{-2+1}}{-2+1}\right|_1^\infty = \left.\frac{x^{-1}}{-1}\right|_1^\infty \\ &= \left.\frac{-1}{x}\right|_1^\infty = \frac{-1}{\infty} - \frac{-1}{1} = 0 - (-1) = 1\,, \end{aligned}$$

so $g(x)$ satisfies both conditions for being a valid probabiity density. $\square$

c. Since the die is fair, each face has a probability of $\frac{1}{6}$ of coming up when the die is rolled. If the random variable X records the number on the face that comes up, it follows that $P(X = 1) = \frac{1}{6}$, $P(X = 2) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$, and $P(X = 3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$. It follows that the expected value of the number that comes up is:

$$E(X) = 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{3}{6} = \frac{14}{6} = \frac{7}{3} \quad \blacksquare$$

3. Do *one (1)* of **a** or **b**. *[10]*

a. The continuous random variable X has an exponential distribution with $\lambda = 1$. Let A be the event that $X \leq \ln(3)$ and B be the event that $\ln(2) \leq X \leq \ln(4)$. Determine whether A and B are independent or not. [Recall that $e^{\ln(t)} = t$ for all $t > 0$ and that $\ln(a) < \ln(b)$ whenever $0 < a < b$.]

b. A fair coin is tossed once and then tossed again until it comes up with the same face that came up on on the first toss. Let the random variable Y count the total number of tosses that occur in this experiment. Find the probability funtion of Y and compute the expected value, $E(Y)$, of Y.

SOLUTIONS. **a.** Recall that the exponential distribution with $\lambda = 1$ has density function $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$. Note that A is basically the interval $(-\infty, \ln(3)]$ and B the interval $[\ln(2), \ln(4)]$, so $A \cap B$ is the interval $[\ln(2), \ln(3)]$. Note also that the antiderivative of e^{-x} is $-e^{-x}$. (This can be done using the substitution $u = -x$, so $du = (-1)\,dx$ and thus $dx = (-1)\,du$. It follows that $\int e^{-x}\,du = \int e^u\,(-1)\,du = (-1)\int e^u\,du = (-1)e^u = -e^{-x}$.) Then

$$\begin{aligned}
P(A) &= \int_{-\infty}^{\ln(3)} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\ln(3)} e^{-x}\,dx = 0 + -e^{-x}\Big|_0^{\ln(3)} \\
&= \left[-e^{-\ln(3)}\right] - \left[-e^{-0}\right] = \left[-\frac{1}{e^{\ln(3)}}\right] - [-1] = -\frac{1}{3} + 1 = \frac{2}{3}, \\
P(B) &= \int_{\ln(2}^{\ln(4)} f(x)\,dx = \int_{\ln(2}^{\ln(4)} e^{-x}\,dx = -e^{-x}\Big|_{\ln(2}^{\ln(4)} \\
&= \left[-e^{-\ln(4)}\right] - \left[-e^{-\ln(2)}\right] = \left[-\frac{1}{e^{\ln(4)}}\right] - \left[-\frac{1}{e^{\ln(2)}}\right] = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}, \text{ and} \\
P(A \cap B) &= \int_{\ln(2}^{\ln(3)} f(x)\,dx = \int_{\ln(2}^{\ln(3)} e^{-x}\,dx = -e^{-x}\Big|_{\ln(2)}^{\ln(3)} \\
&= \left[-e^{-\ln(3)}\right] - \left[-e^{-\ln(2)}\right] = \left[-\frac{1}{e^{\ln(3)}}\right] - \left[-\frac{1}{e^{\ln(2)}}\right] = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6},
\end{aligned}$$

so $P(A \cap B) = \frac{1}{1}6 = \frac{2}{12} = \frac{2}{3} \cdot \frac{1}{4} = P(A) \cdot P(B)$. Hence A and B are independent. $\square$

b. The first toss determines which face we will subsequently be stopping on. Whether it comes up H or T, the number of the rest of the tosses follows a geometric distribution with $p = q = \frac{1}{2}$ because the coin is fair. It follows that $Y - 1$ has a geometric distribution with $p = q = \frac{1}{2}$, which has the probability function $m(k) = q^{k-1}p = \left(\frac{1}{2}\right)^{k-1}\frac{1}{2} = \left(\frac{1}{2}\right)^k$ for $k \geq 1$ and expected value $\frac{1}{p} = \frac{1}{1/2} = 2$. It follows that $Y = (Y-1)+1$ has the probability function $p(n) = m(n-1) = \left(\frac{1}{2}\right)^{n-1}$ (for $n \geq 2$; $m(n) = 0$ otherwise) and has expected value $E(Y) = E(Y - 1 + 1) = E(Y - 1) + 1 = 2 + 1 = 3$. $\blacksquare$

[Total = 30]