Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 2017

Solutions to Assignment #3
A Random Walk

A fair four-sided die has its sides labelled U, D, L, and R, respectively. A token is placed at $(0,0)$ on the Cartesian plane and the die is then rolled repeatedly. After each roll, the token is moved as follows:

Roll	Move
U	$(a,b) \to (a,b+1)$
D	$(a,b) \to (a,b-1)$
L	$(a,b) \to (a+1,b)$
R	$(a,b) \to (a-1,b)$

Let the random variable Y_n be the *taxicab distance** the token is from $(0,0)$ after $n \geq 0$ rolls and the consequent moves. It should be pretty obvious that $Y_0 = 0$: the token starts at $(0,0)$ and $n = 0$ moves have taken place. After that it gets more interesting ...

APOLOGY: *Interesting, indeed. Here I must confess that I screwed up the setup: since horizontal motion takes place when vertical motion does not, and vice versa, horizontal motion and vertical motion are not independent. It follows, for example, that the variances of horizontal distance and of vertical distance cannot be assumed to add up to the variance of their sum, the taxicab distance, as I intended. The setup should have been something like flipping two fair coins simultaneously, one with faces U and D and one with faces L and R, and moving the token accordingly both vertically and horizontally each time. I thought I'd get clever and it backfired in making the problem much, much harder. I grovel in apology. Those who assumed independence will not be penalized for it; those who did not and met the complications that lack of independence causes will also not be penalized.*

1. What is $E(Y_n)$? Explain why as best you can. *[5]*

SOLUTION. First, let us try to figure out the expected location after n rolls and the consequent moves.

Let (H_n, V_n) be the location of the token after n moves. (Note that $Y_n = |H_n| + |V_n|$.) From our setup, we have that $H_0 = V_0 = 0$. Consider the expected values of H_1 and V_1:

$$\begin{aligned} E(H_1) &= 0 \cdot P(U) + 0 \cdot P(D) + 1 \cdot P(L) + (-1) \cdot P(R) \\ &= 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + (-1) \cdot \frac{1}{4} = 0 + 0 + \frac{1}{4} - \frac{1}{4} = 0 \\ E(V_1) &= 1 \cdot P(U) + (-1) \cdot P(D) + 0 \cdot P(L) + 0 \cdot P(R) \\ &= 1 \cdot \frac{1}{4} + (-1) \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{1}{4} - \frac{1}{4} + 0 + 0 = 0 \end{aligned}$$

* The taxicab distance from $(0,0)$ to (a,b) is $|a| + |b|$.

Going from H_1 and V_1 to H_2 and V_2 is just an independent repetition of this experiment, and similarly from move 2 to move 3, and so on. It follows that $E(H_n) = E(nH_1) = nE(H_1) = n \cdot 0 = 0$ and $E(V_n) = E(nV_1) = nE(V_1) = n \cdot 0 = 0$, and so the expected location of the token after n moves is always $(0, 0)$, which has a taxicab distance of 0 from $(0, 0)$ ... If there were any justice here, it would follow that $E(Y_n) = 0$. [Full marks if you reasoned things out along these lines.]

Unfortunately, this is not quite right – darn that lack of justice! Since $Y_n = |H_n|+|V_n|$, $E(Y_n) = E(|H_n| + |V_n|) = E(|H_n|)+E(|V_n|)$, so what we need to do is compute $E(|H_n|)$ and $E(|V_n|)$. Sadly,

$$\begin{aligned} E(|H_1|) &= |0| \cdot P(U) + |0| \cdot P(D) + |1| \cdot P(L) + |-1| \cdot P(R) \\ &= 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = 0 + 0 + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\ E(|V_1|) &= |1| \cdot P(U) + |-1| \cdot P(D) + |0| \cdot P(L) + |0| \cdot P(R) \\ &= 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + 0 + 0 = \frac{1}{2}, \end{aligned}$$

so $E(Y_1) = \frac{1}{2} + \frac{1}{2} = 1$. It is counterintuitive that the expected location is still the starting point, but the expected distance from the starting point is 1, but it makes more sense once you realize that the experiment forces the token to move a distance of 1 from the starting point. It's just the fact that the possible directions and their probabilities balance each other out that makes the expected location still be the origin. It gets worse: I'll leave you to work out what $E(|H_2|)$ and $E(|V_1|)$ are – keep in mind that you will have a $\frac{1}{4}$ chance of moving back to the origin, and a $\frac{3}{4}$ chance of moving a further taxicab step away from it – and nevermind what happens after that ... □

2. What is $V(Y_n)$? Explain why as best you can. *[5]*

Solution. Proceeding most optimistically in the spirit of the instructor's original pious intention as in the first attempt at computating $E(Y_n)$ above, we compute the variance of H_1 and V_1:

$$\begin{aligned} E\left(H_1^2\right) &= 0^2 \cdot P(U) + 0^2 \cdot P(D) + 1^2 \cdot P(L) + (-1)^2 \cdot P(R) \\ &= 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = 0 + 0 + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\ V(H_1) &= E\left(H_1^2\right) - [E(H_1)] = \frac{1}{2} - 0^2 = \frac{1}{2} \\ E\left(V_1^2\right) &= 1^2 \cdot P(U) + (-1)^2 \cdot P(D) + 0^2 \cdot P(L) + 0^2 \cdot P(R) \\ &= 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + 0 + 0 = \frac{1}{2} \\ V(V_1) &= E\left(V_1^2\right) - [E(V_1)] = \frac{1}{2} - 0^2 = \frac{1}{2} \end{aligned}$$

If – we're being optimistic – H_1 and V_1 were independent, or at least had $\text{Cov}(H_1, V_1) = 0$, we could hope that the variance of Y_1 would be the sum of the variances of H_1 and V_1, which would make $V(Y_1) = \frac{1}{2} + \frac{1}{2} = 1$. Since each move is independent, this would allow us to conclude that $V(Y_n) = V(nY_1) = n^2 V(Y_1) = n^2 \cdot 1 = n^2$.

As was noted above, though, H_1 and V_1 are obviously *not* independent, so we could only save the reasoning above if $\text{Cov}(H_1, V_1) = 0$. Let's see if it is:

$$\begin{aligned} E(H_1 V_1) &= 0 \cdot 1 \cdot P(U) + 0 \cdot (-1) \cdot P(D) + 1 \cdot 0 \cdot P(L) + (-1) \cdot 0 \cdot P(R) \\ &= 0 \cdot 1 \cdot \frac{1}{4} + 0 \cdot (-1) \cdot \frac{1}{4} + 1 \cdot 0 \cdot \frac{1}{4} + (-1) \cdot 0 \cdot \frac{1}{4} = 0 + 0 + 0 + 0 = 0 \\ \text{Cov}(H_1, V_1) &= E(H_1 V_1) - E(H_1) \cdot E(V_1) = 0 - 0 \cdot 0 = 0 - 0 = 0 \end{aligned}$$

Wonder of wonders, it is! Pity that Y_n isn't just $H_n + V_n$ ...

As $Y_n = |H_n| + |V_n|$, we really need to compute the variances (and covariance) of $|H_n|$ and $|V_n|$. Let's try this for $n = 1$. Fortunately, we do have one small shortcut: since $H_1^2 = |H_1|^2$ and $V_1^2 = |V_1|^2$, $E\left(|H_1|^2\right) = E\left(H_1^2\right) = \frac{1}{2}$ and $E\left(|V_1|^2\right) = E\left(V_1^2\right) = \frac{1}{2}$. It follows that $V(|H_1|) = E\left(|H_1|^2\right) - [E(|H_1|)]^2 = \frac{1}{2} - \left[\frac{1}{2}\right]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ and $V(|V_1|) = E\left(|V_1|^2\right) - [E(|V_1|)]^2 = \frac{1}{2} - \left[\frac{1}{2}\right]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. For the covariance, we'll have to work slightly harder:

$$\begin{aligned} E(|H_1| \cdot |V_1|) &= |0| \cdot |1| \cdot P(U) + |0| \cdot |-1| \cdot P(D) + |1| \cdot |0| \cdot P(L) + |-1| \cdot 0 \cdot P(R) \\ &= 0 \cdot 1 \cdot \frac{1}{4} + 0 \cdot 1 \cdot \frac{1}{4} + 1 \cdot 0 \cdot \frac{1}{4} + 1 \cdot 0 \cdot \frac{1}{4} = 0 + 0 + 0 + 0 = 0 \\ \text{Cov}(H_1, V_1) &= E(H_1 V_1) - E(H_1) \cdot E(V_1) = 0 - 0 \cdot 0 = 0 - 0 = 0 \end{aligned}$$

Since the covariance is 0 here as well, it follows – even lacking independence – that $V(Y_1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. If, optimistically, we assumed that $V(Y_n) = V(nY_1)$, then we'd have $V(Y_n) = V(nY_1) = n^2 V(Y_1) = \dfrac{n^2}{2}$. I'll leave it to you to figure out just how optimistic this assumption is. ■