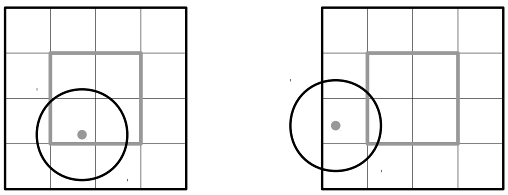
Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 20175

Solutions to Assignment #2
Probable areas?

1. You toss a circular coin randomly onto the Adversary's† infinite checkerboard. That is, it is as likely to turn up in one location as any other location. The coin's diameter is exactly half the length of the sides of the squares of the checkerboard, and the coin does not come to rest on its edge. If the coin touches a side of one or more of the squares, the Adversary keeps it; if it does not, the Adversary returns the coin and gives you three more coins. Is this a fair game? Explain why or why not. *[4]*

SOLUTION. This is really an expected value problem: if on average, you win just as many coins as you lose, the game is fair, and otherwise either you or the Adversary has an edge. The first step, therefore, is to determine what the probabilities are for the coin to touch a side of a square or not. Note that the centre of the coin is as likely to land at one point in a square as it is to land at any other point in that square, and the coin will touch the edge of the square if the centre of the coin is within one radius of the side of the square.



Since the diameter of the coin is half the length of a side of a square, the radius of the coin is one quarter of the length of a side of the square. This means that if we divide the square into sixteen sub-squares of equal size, as in the diagram above, then the coin will avoid touching an edge of the square exactly when its centre lands inside one of the four centre sub-squares. It follows that the probability that the coin will not touch an edge is:

$$P(\text{not touch}) = \frac{\text{area of 4 sub-squares}}{\text{area of square}} = \frac{\text{area of 4 sub-squares}}{\text{area of 16 sub-squares}} = \frac{4}{16} = \frac{1}{4} = 0.25$$

It also follows that the probability that the coin will touch an edge is:

$$P(\text{touch}) = 1 - P(\text{not touch}) = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$$

If one loses a coin when the coin touches an edge but gains three when it does not, the expected number, $E(X)$, of coins one receives when playing the game is:

$$E(X) = (-1) \cdot P(\text{touch}) + 3 \cdot P(\text{not touch}) = (-1) \cdot \frac{3}{4} + 3 \cdot \frac{1}{4} = -\frac{3}{4} + \frac{3}{4} = 0$$

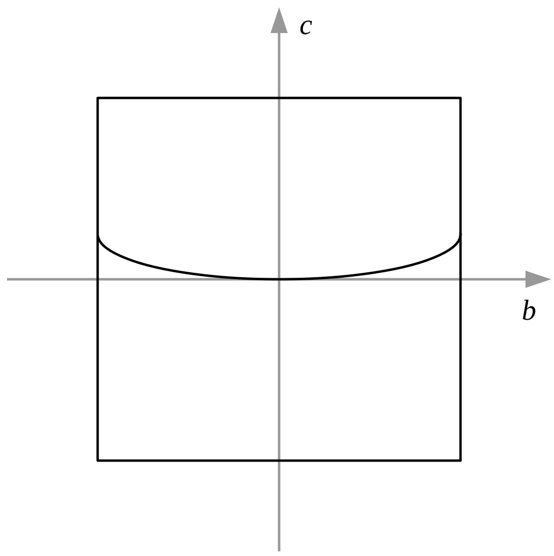
This means the game is fair: one average one would win just as many coins as one would lose. □

† No, not Phil of the Pitchspoon.

2. Suppose two real numbers, b and c, are randomly chosen from the the interval $[-1, 1]$. In each case, any real number in the interval is as likely to be chosen as any other, and it is possible for the two numbers to be equal. What is the probability that the equation $x^2 + bx + c = 0$ has two real solutions? *[6]*

SOLUTION. The quadratic formula tells us that the solutions to the equation $x^2 + bx + c = 0$ are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4 \cdot 1 \cdot c}}{2 \cdot 1} = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$. Inspecting this, it is fairly obvious that there are two distinct real solutions when $b^2 - 4c > 0$, namely $\frac{-b+\sqrt{b^2-4c}}{2}$ and $\frac{-b-\sqrt{b^2-4c}}{2}$, a single real solution when $b^2 - 4c = 0$, namely $-\frac{b}{2}$, and no real solutions if $b^2 - 4c < 0$ (since negative numbers do not have square roots that are real numbers).

Given that b and c are chosen randomly from the interval $[-1, 1]$, the probability that the equation $x^2 + bx + c = 0$ has two real solutions boils down to finding the fraction of the square given by $-1 \le b \le 1$ and $-1 \le c \le 1$ for which $b^2 > 4c$. Note that $b^2 = 4c$ gives the parabola $c = \frac{b^2}{4}$, and that when the point (b, c) is above this parabola, we have $c > \frac{b^2}{4}$ (so $b^2 < 4c$), and when (b, c) is below the parabola, we have $c < \frac{b^2}{4}$ (so $b^2 > 4c$).



The square given by $-1 \le b \le 1$ and $-1 \le c \le 1$ has sides of length $1 - (-1) = 2$, so it has area $2^2 = 4$. The portion of the square for which $b^2 > 4c$ consists of the half of the square below $c = 0$, which has area 2, plus the region between the parabola $c = \frac{b^2}{4}$ and the b-axis (*i.e.* $c = 0$), which has area given by:

$$\int_{-1}^{1} \frac{b^2}{4}\, db = \frac{1}{4}\int_{-1}^{1} b^2\, db = \frac{1}{4} \cdot \frac{b^3}{3}\Big|_{-1}^{1} = \frac{b^3}{12}\Big|_{-1}^{1} = \frac{1^3}{12} - \frac{(-1)^3}{12} = \frac{1}{12} - \frac{-1}{12} = \frac{2}{12} = \frac{1}{6}$$

Thus the total area of the portion of the square for which $b^2 > 4c$ is $2 + \frac{1}{6} = \frac{12}{6} + \frac{1}{6} = \frac{13}{6}$. It follows that the probability hat the equation $x^2 + bx + c = 0$ has two real solutions if b and c are chosen randomly from $[-1, 1]$ is:

$$P(\text{2 real solutions}) = \frac{\text{area below } c = \frac{b^2}{4}}{\text{area of square}} = \frac{13/6}{4} = \frac{13}{6 \cdot 4} = \frac{13}{24} \approx 0.5417 \qquad \blacksquare$$