Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 20175

Solutions to Assignment #1
(Un)biased?

1. Suppose all you have is a biased coin that when tossed comes up heads 41% of the time, tails 58% of the time, and lands on edge 1% of the time. How can you use it to simulate a fair coin, *i.e.* one that comes up heads 50% of the time and tails 50% of the time? Explain why your method works. *[3]*

SOLUTION. Make two tosses of the biased coin at a time. If the two tosses come up HT, record it as a head for the fair coin being simulated; if the two tosses come up TH, record it as a tail for the fair coin being simulated; if the two tosses come up HH, TT, or anything involving E [edge], then toss the biased coin twice again. Repeat as necessary until the two tosses come up as HT or TH.

This works for two reasons. First, $P(HT) = \frac{41}{100} \cdot \frac{58}{100} = \frac{58}{100} \cdot \frac{41}{100} = P(TH)$, so the simulated coin is as likely to come up heads as it is to come up tails. Thus the simulated coin is fair.

Second, the probability that the process will fail to eventually generate a head or tail for the simulated fair coin is 0. Note that the probability of failing to get HT or TH on two successive tosses of the biased coin is :

$$\begin{aligned} 1 - P(HT \text{ or } TH) &= 1 - (P(HT) + P(TH)) = 1 - \left(\frac{41}{100} \cdot \frac{58}{100} + \frac{58}{100} \cdot \frac{41}{100} \right) \\ &= 1 - \frac{4756}{10000} = \frac{5244}{10000} = 0.5244 \end{aligned}$$

It follows that the probability that the process will fail because it goes on forever is:

$$0.5244 \cdot 0.5244 \cdot 0.5244 \cdot \cdots = \lim_{n \to \infty} (0.5244)^n = 0 \qquad \blacksquare$$

2. Suppose all you have is a fair coin. How can you use it to simulate a biased coin such as the one in problem **1** that comes up heads 41% of the time, tails 58% of the time, and lands on edge 1% of the time? Explain why your method works. *[2]*

SOLUTION. Make seven tosses of the fair coin at a time. There are 128 possible outcomes [H or T on each of 7 tosses, for a total of $2^7 = 128$ possibilities]. Assign 41 of these outcomes to record a head for the simulated biased coin, assign 58 more of the outcomes to record a tail for the simulated biased coin, and assign 1 more outcome to record an edge for the simulated biased coin; if any of the remaining 28 outcomes occur, make seven more tosses of the fair coin. Repeat as necessary until one of the 100 assigned outcomes occurs.

This works works for reasons pretty similar to those in the solution to **1**. First, the $100 = 41 + 58 + 1$ assigned outcomes for even tosses of the fair coin are each equally likely. That means that when one of the assigned outcomes occurs, there is a probability of $\frac{41}{100}$ that it was assigned to a head of the simulated coin, a probability of $\frac{58}{100}$ that it was assigned to a tail of the simulated coin, and a probability of $\frac{1}{100}$ that it was assigned to the simulated coin landing on edge.

Second, the probability that the process will fail to eventually generate a head or tail for the simulated fair coin is 0. Note that the probability of failing to get one of the 100 assigned outcomes in seven tosses of the fair coin (*i.e.* getting one of the $128 - 100 = 28$ unassigned outcomes) is $\frac{28}{100}$. It follows that the probability that the process will fail because it goes on forever is:

$$\frac{28}{100} \cdot \frac{28}{100} \cdot \frac{28}{100} \cdot \dots = \lim_{n\to\infty} \left(\frac{28}{100}\right)^n = 0 \qquad \blacksquare$$

3. Suppose all you have is a coin with an unknown bias, but which has a thin enough edge so that it always lands either head or tails when tossed, and will land on each of heads and tails some of the time. How can you use it to simulate a biased coin such as the one in problem **1** that comes up heads 41% of the time, tails 58% of the time, and lands on edge 1% of the time? Explain why your method works. *[2]*

Solution. We first use the coin with an unknown bias to simulate a fair coin by using the procedure given in the solution to **1**. Make two tosses of the biased coin at a time. If the two tosses come up HT, record it as a head for the fair coin being simulated; if the two tosses come up TH, record it as a tail for the fair coin being simulated; if the two tosses come up HH or TT, then toss the biased coin twice again. Repeat as necessary until the two tosses come up as HT or TH. This works for the reasons given in the solution to **1** (with obvious minor modifications due to the leack of an edge, *etc.*).

Now use the simulated fair coin to simulate the desired biased coin, just as in the solution to **2**, which works for the reasons given there ... $\blacksquare$

4. Suppose all you have is a fair coin. How can you use it to – completely accurately! – simulate a biased coin that has $P(H) = \frac{1}{\pi}$ and $P(T) = 1 - \frac{1}{\pi}$? Explain why your method works, or explain why there can be no such method. *[3]*

Note: Keep in mind that $\frac{1}{\pi}$ is irrational (because π is, and so cannot be written as a ratio of integers. That makes this problem harder than **2** and will probably require a different method, if there is any to be had ...

Solution. We will need an expansion of $\frac{1}{\pi}$ in base 2. This is like a base 10 (*i.e.* decimal) expansion, but done in terms of powers of 2 using the digits 0 and 1 instead of powers of 10 and the digits 0, 1, 2, ... , 9. Just as we have, in base 10,

$$\frac{1}{\pi} = 0.31830988\cdots = \frac{3}{10} + \frac{1}{10^2} + \frac{8}{10^3} + \frac{3}{10^4} + \frac{0}{10^5} + \frac{9}{10^6} + \frac{8}{10^7} + \frac{8}{10^8} + \cdots,$$

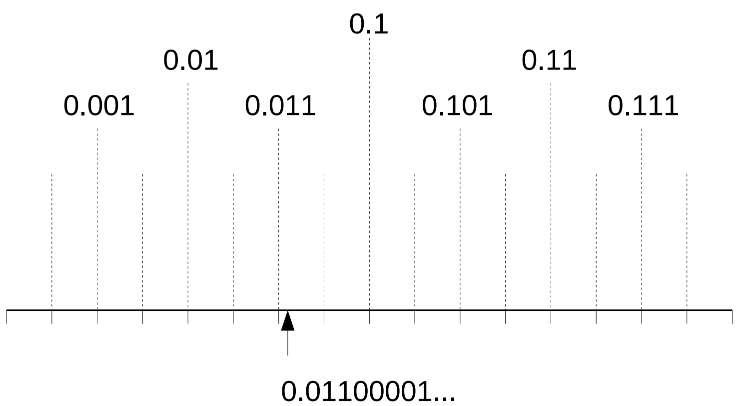
in base 2 we have

$$\frac{1}{\pi} = 0.01100001\cdots = \frac{0}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{0}{2^4} + \frac{0}{2^5} + \frac{0}{2^6} + \frac{0}{2^7} + \frac{1}{2^8} + \cdots$$

Since $\frac{1}{\pi}$ is irrational, both expansions are infinite and non-repeating. We will assume that we have the entire base 2 expansion of $\frac{1}{\pi}$ available. (More practically, one can compute as much of it as one needs, given sufficient time and tedium.)

The simulation process works as follows. Assign the digit 1 to the face H of the fair coin and the digit 0 to the face T. Toss the fair coin until first time the sequence of 0s and 1s it gives differs from the base 2 expansion of $\frac{1}{\pi}$. If the differing digit given by the coin is a 0 and the corresponding digit of the base 2 expansion of $\frac{1}{\pi}$ is 1, we have a head of the biased coin being simulated; if the differing digit given by the coin is a 1 and the corresponding digit of the base 2 expansion of $\frac{1}{\pi}$ is 0, we have a tail of the biased coin being simulated.

Why does this do the job? If we tossed the fair coin infinitely many times, it would give the base 2 expansion of some real number in the interval $[0,1]$. The first digit of the expansion would tell us if it's in the left (if the digit is 0) or right (if the digit is 1) half of the interval $[0,1]$, the second digit tells us if it's in the left or right half of whichever half the first digit put it into, the third digit tells us if it's in the left or right half of that half of a half, and so on. Since the coin is fair, the number so generated is therefore as likely to be in any location in the interval $[0,1]$ as it is in any other. It therefore has a probability of ending up in the subinterval $\left[0,\frac{1}{\pi}\right]$, as opposed to ending up in the subinterval $\left[\frac{1}{\pi},0\right]$, that is proportional to the length of these subintervals. That is, the number has a probability of $\frac{1}{\pi}-0=\frac{1}{\pi}$ of ending up in $\left[0,\frac{1}{\pi}\right]$, and a probability of $1-\frac{1}{\pi}$ of ending up in $\left[\frac{1}{\pi},0\right]$.



Our simulation process generates just enough of this random real number to determine which subinterval it ends up in.

The probability that the random real generated by tossing the fair coin will be (the base 2 expansion) of $\frac{1}{\pi}$ (*i.e.* we'll be tossing the fair coin forever) is 0, by an argument similar to that used in the solution to **1**. ■