Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 2017

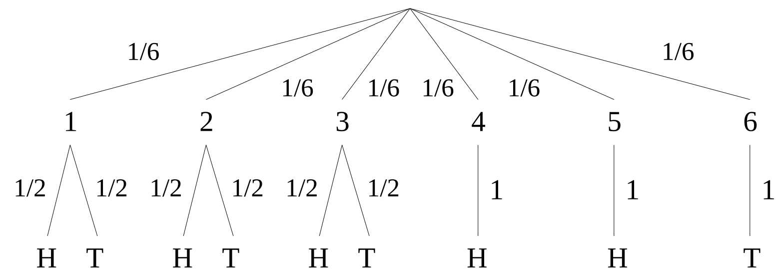
Solutions to the Quizzes

Quiz #1. Thursday, 19 January, 2017. *[12 minutes]*

A standard die is rolled once. If it comes up 1, 2, or 3, you are given a fair coin and toss it once; if the die comes up 4 or 5, you are given a two-headed coin and toss it once; and if it comes up 6, you are given a two-tailed coin and toss it once.

1. Draw the tree diagram for this experiment. *[2]*
2. What is the sample space Ω for this experiment? *[1.5]*
3. Determine the probability that a head came up on the coin toss in the second stage of the experiment. *[1.5]*

SOLUTION. 1. Here is the complete tree diagram for this experiment:



□

2. $\Omega = \{ 1H, 1T, 2H, 2T, 3H, 3T, 4H, 5H, 6T \}$ □

3. If A is the event that a head came up on the coin toss in the second stage of the experiment, then, as a collection of outcomes, $A = \{ 1H, 2H, 3H, 4H, 5H \}$. Thus, getting the probabilities of the individual outcomes from the tree diagram given in the solution to question 1 above, we have:

$$\begin{aligned} P(A) &= m(1H) + m(2H) + m(3H) + m(4H) + m(5H) \\ &= \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 1 \\ &= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{2}{12} + \frac{2}{12} = \frac{7}{12} \quad \blacksquare \end{aligned}$$

Quiz #2. Thursday, 26 January, 2017. *[10 minutes]*

A hand of five cards is drawn simultaneously (and randomly!) from a standard 52-card deck.

1. What is the probability that the hand has exactly two cards of the same kind? *[5]*

SOLUTION. First, there are $\binom{52}{5} = \frac{52!}{(52-5)!5!} = \frac{52 \cdot 52 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{311875200}{120} = 2598960$ ways to pick out 5 cards out of 52 cards if order doesn't matter. Note that each of these hands is as likely as any other to be drawn.

Second, there are $\binom{13}{1} = 13$ ways to choose the kind for the pair and $\binom{4}{2} = 6$ ways to choose a pair from the four cards of that kind in the deck. Similarly, there are $\binom{12}{3} = \frac{12!}{(12-3)!3!} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = \frac{1320}{6} = 220$ ways to pick three more kinds for the remaining cards in the hand, and, for each of those three kinds, $\binom{4}{1} = 4$ ways to pick a card of that kind. It follows that there are

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1} = 13 \cdot 6 \cdot 220 \cdot 4 \cdot 4 \cdot 4 = 1098240$$

possible hands with exactly two cards of the same kind.

Finally, since each hand is alikely to be drawn as any other, it follows that:

$$P(\text{exactly 2 cards of the same kind}) = \frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{5}} = \frac{1098240}{2598960} \approx 0.42$$

An unsimplified answer is just fine for this quiz, but in the real world you would probably want the decimal, quite possibly to more places than given here. ■

Quiz #3. Thursday, 2 February, 2017. *[10 minutes]*

A fair coin is tossed four times. Let A be the event that the second toss was a tail and let B be the event that there were exactly two tails among the four tosses.

1. Compute $P(A|B)$, the conditional probability of A given B. *[4]*
2. Determine whether the events A and B are independent. *[1]*

SOLUTION. 1. We need to compute $P(B)$ and $P(A \cap B)$ in order to compute $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Note that if a fair coin is tossed four times, there are $2^4 = 16$ equally likely outcomes.

First, there are $\binom{4}{2} = 6$ outcomes in B, as this is the number of ways one could choose exactly two of the four tosses to be those in which a tail occurred. This means that $P(B) = \frac{6}{16}$.

Second, $A \cap B$ consists of those outcomes in which a tail occurred on the second toss and exactly one tail occurred among the other three tosses. There are $\binom{3}{1} = 3$ such outcomes as there are three other tosses from which to choose the one other toss in which a tail could occur. This means that $P(A \cap B) = \frac{3}{16}$.

It follows that $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{3/16}{6/16} = \frac{3}{6} = \frac{1}{2}$. □

2. Since the coin is fair, the probability that it comes up tails on any particular toss is $\frac{1}{2}$, so, in particular, $P(A) = \frac{1}{2}$. Since $P(A|B) = \frac{1}{2} = P(A)$, the events A and B are independent. ■

Quiz #4. Thursday, 9 February, 2017. *[10 minutes]*

A fair standard six-sided die is rolled four times.

1. What is the expected value of the number of rolls in which the die comes up with a multiple of three? *[5]*

SOLUTION. When the fair die is rolled once, each of the six faces is a as likely to come up as any other, so each has a probability of $\frac{1}{6}$ of coming up. The multiples of three among the integers 1 through 6 are 3 and 6, so it follows that

$$P(\text{multiple of 3}) = m(3) + m(6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}\,.$$

Rolling the die four times and counting the rolls in which a multiple of 3 came up is like repeating a Bernoulli trial with a probability of success of $\frac{1}{3}$ four times. This has a binomial distribution with $n = 4$, $p = \frac{1}{3}$, and $q = 1 - \frac{1}{3} = \frac{2}{3}$, so it has an expected value of $E(X) = np = 4 \cdot \frac{1}{3} = \frac{4}{3}$. ■

Quiz #5. Thursday, 16 February, 2017. *[12 minutes]*

The blades of grass on a certain lawn are all between 3.5 and 6.5 *cm* long. The random variable X is the length of randomly chosen blade of grass from this lawn, and is as likely to have any given value between 3.5 and 6.5 *cm* as it is to have any other value between these extremes.

1. What is the probability density function of X? *[2]*
2. Verify that the function you gave to answer question 1 is a valid probability density. *[2]*
3. Find the expected value, $E(X)$, of X. *[1]*

SOLUTION. 1. Denote the probability density function of X by $f(x)$. Since all the values that X takes on are between 3.5 and 6.5, we must have $f(x) = 0$ when $x < 3.5$ or $x > 6.5$; since we also have that any value between 3.5 and 6.5 is equally likely, X has a continuous uniform distribution on $[3.5, 6.5]$, so for $3.5 \leq x \leq 6.5$ we have $f(x) = \dfrac{1}{6.5 - 3.5} = \dfrac{1}{3}$. Thus the probability density function of X is $f(x) = \begin{cases} \frac{1}{3} & 3.5 \leq x \leq 6.5 \\ 0 & x < 3.5 \text{ or } x > 6.5 \end{cases}$. □

2. First, $f(x) \geq 0$ for all x by its definition: $f(x) = \dfrac{1}{3} > 0$ for $3.5 \leq x \leq 6.5$, and $f(x) = 0$ otherwise. Second,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{infty}^{3.5} 0\,dx + \int_{3.5}^{6.5} \frac{1}{3}\,dx + \int_{6.5}^{\infty} 0\,dx = 0 + \frac{1}{3}x\Big|_{3.5}^{6.5} + 0 \\ &= \frac{1}{3} \cdot 6.5 - \frac{1}{3} \cdot 3.5 = \frac{6.5 - 3.5}{3} = \frac{3}{3} = 1\,, \end{aligned}$$

so $f(x)$ meets both requirements for being a valid probability density. □

3. As noted in class, the expected value of a continuous random variable with a uniform distribution on an interval $[a, b]$ has expected value $\dfrac{a+b}{2}$. Since, as observed in the solution to question 1 above, the given random variable X has uniform distribution on $[3.5, 6.5]$, its expected value is $E(X) = \dfrac{3.5 + 6.5}{2} = \dfrac{10}{2} = 5$. ■

Quiz #6. Thursday, 9 March, 2017. *[10 minutes]*

A fair four-sided die has its faces labelled with the numbers -1, -2, $+3$, and $+4$. It is rolled once and the random variable X returns the number on the face that comes up.

1. Find the probability function of X. *[1]*
2. Compute the expected value $E(X)$ of X. *[2]*
3. Compute the variance $V(X)$ of X. *[2]*

SOLUTION. 1. Since the die is fair, each of its four faces is as likely to come up as any other, giving X a discrete uniform distribution, *i.e.* $m(-1) = m(-2) = m(+3) = m(+4) = \frac{1}{4} = 0.25$. $\square$

2. By definition:

$$E(X) = \sum_x xm(x) = (-1)\cdot\frac{1}{4} + (-2)\cdot\frac{1}{4} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{4} = \frac{-1-2+3+4}{4} = \frac{4}{4} = 1 \quad \square$$

3. We will use the formula $V(X) = E\left(X^2\right) - [E(X)]^2$, so we first need to compute $E\left(X^2\right)$:

$$E\left(X^2\right) = \sum_x x^2 m(x) = (-1)^2\cdot\frac{1}{4} + (-2)^2\cdot\frac{1}{4} + 3^2\cdot\frac{1}{4} + 4^2\cdot\frac{1}{4} = \frac{1+4+9+16}{4} = \frac{30}{4} = \frac{15}{2}$$

It follows that $V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{15}{2} - 1 = \frac{15}{2} - \frac{2}{2} = \frac{13}{2} = 6.5$. $\blacksquare$

Quiz #7. ~~Thursday, 16~~ Tuesday, 21 March, 2017. *[10 minutes]*

Answer *one* (1) of the following questions.

1. Show that $E\left(X^2\right) \geq [E(X)]^2$ for any random variable X. *[5]*
2. Suppose X_1, X_2, ... , X_9 are independent and identically distributed random variables, with $E(X_i) = 41$ and variance $V(X_i) = 81$ for each i. Explain as fully as you can why $\frac{X_1+X_2+\cdots+X_9}{9}$ is likely to be closer to 41 than X_1 is by itself. *[5]*

SOLUTION. 1. The very short, but good enough, version is that $V(X) = E\left(X^2\right) - [E(X)]^2$ and – as noted in class – is always positive, which is only possible if $E\left(X^2\right) \geq [E(X)]^2$.

The complete to the point of paranoia version is that $[X - E(X)]^2 \geq 0$ because it is a square, from which it follows that $V(X) = E\left([X - E(X)]^2\right) \geq 0$ too. The computation given in class, exploiting the linearity of expected value and the fact that $E(X)$ is a constant,

$$\begin{aligned} V(X) &= E\left([X - E(X)]^2\right) = E\left(X^2 - 2E(X)\cdot X + [E(X)]^2\right) \\ &= E\left(X^2\right) - E\left(2E(X)\cdot X\right) + E\left([E(X)]^2\right) = E\left(X^2\right) - 2E(X)\cdot E(X) + [E(X)]^2 \\ &= E\left(X^2\right) - 2\,[E(X)]^2 + [E(X)]^2 = E\left(X^2\right) - [E(X)]^2 \ , \end{aligned}$$

then implies that $E\left(X^2\right) - [E(X)]^2 \geq 0$, so $E\left(X^2\right) \geq [E(X)]^2$. $\square$

2. As noted in class, $\frac{X_1+X_2+\cdots+X_9}{9}$ has the same expected value as any of the identically distributed X_i. Observe that because the X_i are independent, the properties of variance tell us that:

$$\begin{aligned} V\left(\frac{X_1+X_2+\cdots+X_9}{9}\right) &= \left(\frac{1}{9}\right) V\left(X_1+X_2+\cdots+X_9\right) \\ &= \frac{1}{81}\left[V(X_1) + V(X_2) + \cdots + V(X_9)\right] \\ &= \frac{1}{81}\left[81+81+\cdots+81\right] = \frac{1}{81}[9\cdot 81] = 9 \end{aligned}$$

Thus $V\left(\frac{X_1+X_2+\cdots+X_9}{9}\right) = 9 < 81 = V(X_1)$, and since variance is a measure of how close to its expected value a random variable is likely to end up, $\frac{X_1+X_2+\cdots+X_9}{9}$ is likely to be closer to 41 than X_1 is by itself. $\blacksquare$

Quiz #8. Thursday, 23 March, 2017. *[10 minutes]*

1. How many distinguishable ways are there to take all the letters, including the repetitions, in the word "Peterborough" and arrange them in a row? *[5]*

SOLUTION. "Peterborough" has twelve letters, counting the repetitions: there are two "e"s, two "o"s, and two "r"s. There are 12! ways to arrange twelve distinct things, but for each of the three identical pairs of letters there are 2! ways to rearrange them among themselves without the change being apparent. It follows that there are $\dfrac{12!}{2!2!2!} = 59875200$ distinguishable ways to arrange all the letters in the word "Peterborough". ■

Quiz #9. Thursday, 30 March, 2017. *[10 minutes]*

Suppose X is a random variable with $\mu = E(X) = 1$ and $\sigma = \sqrt{V(X)} = 2$.

1. What can you deduce about $P(|X-1| \geq 4)$ using Chebyshev's Inequality if you know nothing else about X? *[2]*
2. What can you deduce about $P(X \geq 5)$ using Chebyshev's Inequality if you also know that X is symmetric about $\mu = 1$ (*i.e.* $P(X-1 \geq a) = P(X-1 \leq -a)$ for all a)? *[1.5]*
3. What can you deduce about $P(X \geq 5)$ using Chebyshev's Inequality if instead you know that X is never negative? *[1.5]*

SOLUTION. 1. Chebyshev's Inequality states that $P(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$. We have $\mu = 1$ and $\sigma = 2$, so looking at $P(|X-1| \geq 4)$ we see that we must have $t = 4$. Thus $P(|X-1| \geq 4) \leq \frac{2^2}{4^2} \leq \frac{4}{16} = \frac{1}{4} = 0.25$. □

2. First, note that $P(X \geq 5) = P(X-1 \geq 4)$. Second, because X is symmetric about $\mu = 1$, we have $P(|X-1| \geq 4) = P(X-1 \geq 4) + P(X-1 \leq -4) = 2P(X-1 \geq 4)$. Using the answer to question 1 above, this means that $P(X-1 \geq 4) = \frac{1}{2}P(|X-1| \geq 4) \leq \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. □

3. If X is never negative, then $|X-1| \geq 4$ only when $X-1 \geq 4$, *i.e.* when $X \geq 5$. Thus $P(X \geq 5) = P(X-1 \geq 4) = P(|X-1| \geq 4)$, which, by Chebyshev's Inequality, is $\leq \frac{1}{4} = 0.25$, as in the answer to question 1 above. That is, we get the same answer as in question 1 in this case ... □

Quiz #10. Thursday, 6 April, 2017. *[10 minutes]*

Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	1	2
1	0.1	0.2
2	0.2	0.2
3	0.1	0.2

1. Compute the expected values $E(X)$ and $E(Y)$, the variances $V(X)$ and $V(Y)$, and the covariance $\text{Cov}(X, Y)$ of X and Y. *[5]*

Solution. Here we go:

$$\begin{aligned}
E(X) &= 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3) \\
&= 1(0.1 + 0.2) + 2(0.2 + 0.2) + 3(0.1 + 0.2) = 0.3 + 0.8 + 0.9 = 2 \\
E(Y) &= 1 \cdot P(Y = 1) + 2 \cdot P(Y = 2) = 1(0.1 + 0.2 + 0.1) + 2(0.2 + 0.2 + 0.2) \\
&= 0.4 + 1.2 = 1.6 \\
E\left(X^2\right) &= 1^2(0.1 + 0.2) + 2^2(0.2 + 0.2) + 3^2(0.1 + 0.2) = 1 \cdot 0.3 + 4 \cdot 0.4 + 9 \cdot 0.3 \\
&= 0.3 + 1.6 + 2.7 = 4.6 \\
V(X) &= E\left(X^2\right) - [E(X)]^2 = 4.6 - 2^2 = 4.6 - 4 = 0.6 \\
E\left(Y^2\right) &= 1^2(0.1 + 0.2 + 0.1) + 2^2(0.2 + 0.2 + 0.2) = 1 \cdot 0.4 + 4 \cdot 0.6 \\
&= 0.4 + 2.4 = 2.8 \\
V(Y) &= E\left(Y^2\right) - [E(Y)]^2 = 2.8 - (1.6)^2 = 2.8 - 2.56 = 0.24 \\
E(XY) &= 1 \cdot 1 \cdot P(X = 1 \,\&\, Y = 1) + \cdots + 3 \cdot 2 \cdots P(X = 3 \,\&\, Y = 2) \\
&= 1 \cdot 1 \cdot 0.1 + 1 \cdot 2 \cdot 0.2 + 2 \cdot 1 \cdot 0.2 + 2 \cdot 2 \cdot 0.2 + 3 \cdot 1 \cdot 0.1 + 3 \cdot 2 \cdot 0.2 \\
&= 0.1 + 0.4 + 0.4 + 0.8 + 0.3 + 1.2 = 3.2 \\
\text{Cov}(X, Y) &= E(XY) - E(X) \cdot E(Y) = 3.2 - 2 \cdot 1.6 = 3.2 - 3.2 = 0
\end{aligned}$$

Whew! ■