Mathematics 1550H – Introduction to probability

TRENT UNIVERSITY, Winter 2017

FINAL EXAMINATION

Wednesday, 19 April, 2017

Time-space: 19:00–22:00 in the Gym *Inflicted by* Стефан Біланюк.

Instructions: Do both of parts **X** and **Y**, and, if you wish, part **Z**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one 8.5″ × 11″ or A4 aid sheet; standard normal table; one brain maximum.

Part X. Do all of **1**–**5**. *[Subtotal = 68/100]*

1. A fair three-sided die that has faces numbered 1, 2, and 3, respectively, is tossed twice.

a. Draw the complete tree diagram for this experiment. *[3]*

b. What are the sample space and probability function for this experiment? *[5]*

c. Let the random variable X give the sum of the numbers that come up in the two tosses. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[7]*

2. Let W be a continuous random variable with the following probability density function:

$$g(w) = \begin{cases} \frac{3}{4}\left(1-w^2\right) & -1 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $g(w)$ is indeed a probability density function. *[8]*

b. Compute the probability that $W \leq \frac{1}{2}$, given that $W \geq 0$. *[7]*

3. A hand of five cards is drawn simultaneously from a standard 52-card deck. Let A be the event that the hand is the *Dead Man's Hand* supposedly held by "Wild Bill" Hickok when he was murdered in 1876 — $A\spadesuit$, $A\clubsuit$, $8\spadesuit$, $8\clubsuit$, and $Q\clubsuit$ — and let B be the event that the hand includes two pairs and one more card of a different kind. Compute the conditional probabilities $P(A|B)$ and $P(B|A)$. *[10]*

4. A fair coin is tossed until it comes up heads for the second time. Let Y be the number of tosses required, let A be the event that $Y \leq 5$, and let B be the event that $Y \geq 4$.

a. What are the expected value, $E(Y)$, and variance, $V(Y)$, of Y? *[6]*

b. Compute $P(A|B)$. *[7]*

5. Suppose X is a continuous random variable that has a normal distribution with expected value $\mu = 3$ and variance $\sigma^2 = 4$.

a. Compute $P(1 \leq X \leq 5)$ with the help of a standard normal table. *[6]*

b. Use Chebyshev's Inequality to estimate $P(X \leq 5)$. *[9]*

*[Parts **Y** and **Z** are on page 2.]*

Part Y. Do any *two* (2) of **6**–**9**. *[Subtotal = 32/100]*

6. Let $g(x) = \begin{cases} \frac{1}{2} & 0 \le x \le 1 \\ \frac{1}{2}e^{-x} & x \ge 1 \\ 0 & x < 0 \end{cases}$ be the probability density function of the continuous random variable X.

a. Verify that $g(x)$ is indeed a probability density function. *[6]*

b. Compute the expected value $E(X)$ and variance of $V(X)$ of X. *[10]*

7. A jar contains two identical red balls and three identical blue balls. Balls are drawn randomly from the jar, one at a time and without replacement, until all five balls have been removed from the jar.

a. Find the sample space and probability function for this experiment. *[6]*

b. Let A be the event that the third ball drawn is blue, and let B be the event that the first ball drawn is red. Determine whether A and B are independent. *[5]*

c. The random variable R returns in which draw the second red ball came up. Compute the expected value $E(R)$ and variance $V(R)$ of R *[5]*

8. Suppose X_1 and X_2 are independent continuous random variables that each have a standard normal distribution. Let $X = X_1 + X_2$.

a. Compute the expected value, $E(X)$, and variance, $V(X)$, of X. *[6]*

b. What is the distribution of X? *[10 = 4 if you guess right + 6 if you show it]*

9. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	1	2	3
-1	0.1	0.1	0.2
1	0.1	0.2	0
3	0.2	0	0

a. Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\mathrm{Cov}(X, Y)$ of X and Y. *[12]*

b. Let $W = 2Y - 3X$. Compute $E(W)$ and $V(W)$. *[4]*

[Total = 100]

Part Z. Bonus!

○●. Give as clever — *i.e.* with a simple set-up and an easy, but not obvious, solution — a probability problem as you can. *[1]*

●○. Write an original little poem about probability or mathematics in general. *[1]*

*[Part **X** is on page 1.]*

I HOPE THAT YOU ENJOYED THE COURSE. HAVE A GOOD THE SUMMER!