Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 2017

SOLUTIONS TO THE FINAL EXAMINATION
Wednesday, 19 April, 2017

Time-space: 19:00–22:00 in the Gym *Inflicted by* Стефан Біланюк.

Instructions: Do both of parts **X** and **Y**, and, if you wish, part **Z**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; standard normal table; one brain maximum.

Part X. Do all of **1**–**5**. *[Subtotal = 68/100]*

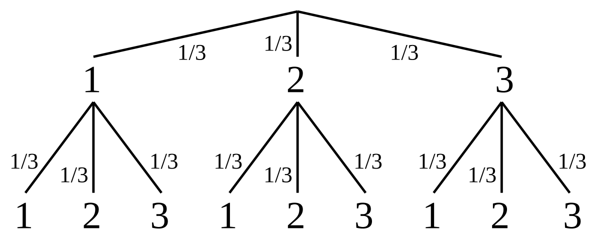
1. A fair three-sided die that has faces numbered 1, 2, and 3, respectively, is tossed twice.

a. Draw the complete tree diagram for this experiment. *[3]*

b. What are the sample space and probability function for this experiment? *[5]*

c. Let the random variable X give the sum of the numbers that come up in the two tosses. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[7]*

SOLUTIONS. **a.** Here it is:



$\square$

b. $\Omega = \{ (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3) \}$. Since the die is fair, each outcome is equally likely, and so we have a uniform distribution with $m(\omega) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$ for each outcome $\omega \in \Omega$. $\square$

c. The possible values of X are $2 = 1+1$, $3 = 1+2 = 2+1$, $4 = 1+3 = 2+2 = 3+1$, $5 = 2+3 = 3+2$, and $6 = 3+3$, and their respective probabilities are given by:

$$\begin{aligned}
P(X=2) &= m(1,1) = \frac{1}{9} \\
P(X=3) &= m(1,2) + m(2,1) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9} \\
P(X=4) &= m(1,3) + m(2,2) + m(3,1) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{3}{9} = \frac{1}{3} \\
P(X=5) &= m(2,3) + m(3,2) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9} \\
P(X=6) &= m(3,3) = \frac{1}{9}
\end{aligned}$$

It follows that

$$E(X) = \sum_{k=2}^{6} k \cdot P(X=k) = 2 \cdot \frac{1}{9} + 3 \cdot \frac{2}{9} + 4 \cdot \frac{3}{9} + 5 \cdot \frac{2}{9} + 6 \cdot \frac{1}{9} = \frac{2+6+12+10+6}{9} = \frac{36}{9} = 4$$

which you could also get by observing that the probability distribution of X is symmetric about 4. Since

$$E\left(X^2\right) = \sum_{k=2}^{6} k^2 \cdot P(X=k) = 2^2 \cdot \frac{1}{9} + 3^2 \cdot \frac{2}{9} + 4^2 \cdot \frac{3}{9} + 5^2 \cdot \frac{2}{9} + 6^2 \cdot \frac{1}{9}$$
$$= \frac{4+18+48+50+36}{9} = \frac{156}{9} = \frac{52}{3} = 17\frac{1}{3} \approx 17.3,$$

we get that $V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{52}{3} - 4^2 = \frac{52}{3} - 16 = \frac{4}{3} \approx 1.3$. ■

2. Let W be a continuous random variable with the following probability density function:

$$g(w) = \begin{cases} \frac{3}{4}\left(1-w^2\right) & -1 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Verify that $g(w)$ is indeed a probability density function. *[8]*

b. Compute the probability that $W \leq \frac{1}{2}$, given that $W \geq 0$. *[7]*

Solutions. **a.** First, observe that when $-1 \leq w \leq 1$, we get $w^2 \leq 1$, and so $g(w) = \frac{3}{4}\left(1-w^2\right) \geq 0$. Since $g(w) = 0$ otherwise, it follows that $g(w) \geq 0$ for all w.

Second, using the Power Rule for integration at the key step,

$$\int_{-\infty}^{\infty} g(w)\,dw = \int_{-\infty}^{-1} 0\,dw + \int_{-1}^{1} \frac{3}{4}\left(1-w^2\right)\,dw + \int_{1}^{\infty} 0\,dw$$
$$= 0 + \frac{3}{4}\int_{-1}^{1}\left(1-w^2\right)\,dw + 0 = \frac{3}{4}\left(w - \frac{w^3}{3}\right)\Big|_{-1}^{1}$$
$$= \frac{3}{4}\left(1 - \frac{1^3}{3}\right) - \frac{3}{4}\left((-1) - \frac{(-1)^3}{3}\right)$$
$$= \frac{3}{4}\left(1 - \frac{1}{3}\right) - \frac{3}{4}\left((-1) - \frac{-1}{3}\right) = \frac{3}{4} \cdot \frac{2}{3} - \frac{3}{4} \cdot \frac{-2}{3}$$
$$= \frac{1}{2} - \frac{-1}{2} = \frac{1}{2} + \frac{1}{2} = 1,$$

so it follows that $g(w)$ is indeed a valid probability density function. □

b. Note that this is a conditional probability problem: the question asks one to work out $P\left(W \leq \frac{1}{2} | W \geq 0\right)$. By definition, $P\left(W \leq \frac{1}{2} | W \geq 0\right) = \frac{P\left(W \leq \frac{1}{2} \text{ and } W \geq 0\right)}{P(W \geq 0)} = \frac{P\left(0 \leq W \leq \frac{1}{2}\right)}{P(W \geq 0)}$, so we need to work out $P\left(0 \leq W \leq \frac{1}{2}\right)$ and $P\left(W \geq 0\right)$:

$$P(W \geq 0) = \int_{0}^{\infty} g(w)\,dw = \int_{0}^{1} \frac{3}{4}\left(1-w^2\right)\,dw + \int_{1}^{\infty} 0\,dw = \frac{3}{4}\int_{0}^{1} \frac{3}{4}\left(1-w^2\right)\,dw + 0$$
$$= \frac{3}{4}\left(w - \frac{w^3}{3}\right)\Big|_{0}^{1} = \frac{3}{4}\left(1 - \frac{1^3}{3}\right) - \frac{3}{4}\left(0 - \frac{0^3}{3}\right) = \frac{3}{4} \cdot \frac{2}{3} - \frac{3}{4} \cdot 0 = \frac{1}{2} - 0 = \frac{1}{2}$$

and

$$P\left(0 \le W \le \frac{1}{2}\right) = \int_0^{1/2} g(w)\,dw = \int_0^{1/2} \frac{3}{4}\left(1-w^2\right)\,dw = \frac{3}{4}\left(w - \frac{w^3}{3}\right)\Big|_0^{1/2}$$
$$= \frac{3}{4}\left(\frac{1}{2} - \frac{(1/2)^3}{3}\right) - \frac{3}{4}\left(0 - \frac{0^3}{3}\right) = \frac{3}{4}\cdot\frac{11}{24} - \frac{3}{4}\cdot 0 = \frac{11}{32}$$

Thus $P\left(W \le \frac{1}{2} | W \ge 0\right) = \dfrac{P\left(0 \le W \le \frac{1}{2}\right)}{P\left(W \ge 0\right)} = \dfrac{11/32}{1/2} = \dfrac{22}{32} = \dfrac{11}{16}$. ■

3. A hand of five cards is drawn simultaneously from a standard 52-card deck. Let A be the event that the hand is the *Dead Man's Hand* supposedly held by "Wild Bill" Hickok when he was murdered in 1876 — $A\spadesuit$, $A\clubsuit$, $8\spadesuit$, $8\clubsuit$, and $Q\clubsuit$ — and let B be the event that the hand includes two pairs and one more card of a different kind. Compute the conditional probabilities $P(A|B)$ and $P(B|A)$. *[10]*

Solution. There are $\binom{52}{5}$ possible hands of five cards that could be drawn from the deck. Just one of these hands is in the event A, so $P(A) = \frac{1}{\binom{52}{5}}$. To find the number of hands in the event B, note that there are $\binom{13}{2}$ ways to choose the kinds for the two pairs, $\binom{4}{2}$ ways for each of these kinds to choose a pair from the four cards of that kind, and $\binom{44}{1}$ ways to pick one more card of some other kind. Thus $P(B) = \frac{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}{\binom{52}{5}}$. Since the only hand in A is also in B, $A \cap B = A$ and $P(A \cap B) = P(A)$. Hence

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/\binom{52}{5}}{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}/\binom{52}{5}} = \frac{1}{\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}}$$

and

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1\,. \qquad ■$$

4. A fair coin is tossed until it comes up heads for the second time. Let Y be the number of tosses required, let A be the event that $Y \le 5$, and let B be the event that $Y \ge 4$.

a. What are the expected value, $E(Y)$, and variance, $V(Y)$, of Y? *[6]*

b. Compute $P(A|B)$. *[7]*

Solutions. **a.** Y has a negative binomial distribution with $p = q = \frac{1}{2}$ and $k = 2$, so it has probability function is $m(y) = \binom{y-1}{2-1}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{y-2} = \binom{y-1}{1}\left(\frac{1}{2}\right)^y$ (for $y \ge 2$; $m(y) = 0$ otherwise), and has expected value $E(Y) = \frac{2}{\frac{1}{2}} = 4$ and variance $V(Y) = \frac{2\cdot\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 4$. □

b. Note that

$$P(B) = P(Y \ge 4) = 1 - P(Y < 4) = 1 - [P(Y=2) + P(Y=3)] = 1 - [m(2) + m(3)]$$
$$= 1 - \left[\binom{2-1}{1}\left(\frac{1}{2}\right)^2 + \binom{3-1}{1}\left(\frac{1}{2}\right)^3\right] = 1 - \left[\frac{1}{4} + \frac{2}{8}\right] = 1 - \frac{1}{2} = \frac{1}{2},$$

and, since $A \cap B$ is the event that $4 \leq Y \leq 5$, *i.e.* that $Y = 4$ or $Y = 3$,

$$\begin{aligned} P(A \cap B) &= P(Y=4) + P(Y=5) = m(4) + m(5) \\ &= \binom{4-1}{1}\left(\frac{1}{2}\right)^4 + \binom{5-1}{1}\left(\frac{1}{2}\right)^5 = \frac{3}{16} + \frac{4}{32} = \frac{5}{16}. \end{aligned}$$

It follows that $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{5}{16}}{\frac{1}{2}} = \dfrac{5}{8}$. ■

5. Suppose X is a continuous random variable that has a normal distribution with expected value $\mu = 3$ and variance $\sigma^2 = 4$.

a. Compute $P(1 \leq X \leq 5)$ with the help of a standard normal table. *[6]*

b. Use Chebyshev's Inequality to estimate $P(X \leq 5)$. *[9]*

SOLUTIONS. **a.** Since X has a normal distribution with expected value $\mu = 3$ and variance $\sigma^2 = 4$ (so $\sigma = \sqrt{4} = 2$), $Z = \frac{X-\mu}{\sigma} = \frac{X-3}{2}$ has a standard normal distribution. Since

$$\begin{aligned} 1 \leq X \leq 5 &\Leftrightarrow -2 = 1-3 \leq X-3 \leq 5-3 = 2 \\ &\Leftrightarrow -\frac{2}{2} \leq \frac{X-3}{2} \leq \frac{2}{2} \Leftrightarrow -1 \leq Z \leq 1, \end{aligned}$$

we get, using the cumulative standard normal table supplied with the exam:

$$\begin{aligned} P(1 \leq X \leq 5) &= P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z < -1) \\ &\approx 0.8413 - 0.1587 = 0.6826 \qquad \square \end{aligned}$$

b. The problem here is to find something suitably related to $P(X \leq 5)$ to which Chebyshev's Inequality can be applied. Observe first that $P(X \leq 5) = 1 - P(X \geq 5) = 1 - P(X - 3 \geq 2)$. Now, since the normal distribution with $\mu = 3$ and $\sigma = 2$ is symmetric about $\mu = 3$, $P(X-3 \geq 2) = P(X-3 \leq -2)$, and since $P(|X-3| \geq 2) = P(X-3 \geq 2) + P(X-3 \leq -2)$, we have that $P(X-3 \geq 2) = \frac{1}{2}P(|X-3| \geq 2)$. By Chebyshev's Inequality, we have that $P(|X-3| \geq 2) \leq \frac{2^2}{\sigma^2} = \frac{2^2}{2^2} = 1$ [We already knew that just because it was a probability, didn't we? So Chebyshev's Inequality wasn't really necessary here ...], so it follows that $P(X-3 \geq 2) = \frac{1}{2}P(|X-3| \geq 2) \leq \frac{1}{2} \cdot 1 = \frac{1}{2}$. Thus

$$P(X \leq 5) = 1 - P(X \geq 5) = 1 - P(X-3 \geq 2) \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Note that $0.6826 \geq 0.5 = \frac{1}{2}$ indeed. ■

[Parts **Y** *and* **Z** *are on page 2.]*

Part Y. Do any *two* (2) of **6–9**. *[Subtotal = 32/100]*

6. Let $g(x) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 1 \\ \frac{1}{2}e^{-x} & x \geq 1 \\ 0 & x < 0 \end{cases}$ be the probability density function of the continuous random variable X.

a. Verify that $g(x)$ is indeed a probability density function. *[6]*

b. Compute the expected value $E(X)$ and variance of $V(X)$ of X. *[10]*

SOLUTIONS. **a.** First, $g(x) \geq 0$ for all x since $0 \geq 0$ for $x < 0$, $\frac{1}{2} \geq 0$ when $0 \leq x \leq 1$, and $\frac{1}{2}e^{-x} \geq 0$ (because $e^t > 0$ for all t) for $x \geq 1$. Second,

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_0^1 \frac{1}{2}\,dx + \int_1^{\infty} \frac{1}{2}e^{-x}\,dx = 0 + \frac{1}{2}x\Big|_0^1 + \frac{1}{2}\left(-e^{-x}\right)\Big|_1^{\infty} \\
&= \left[\frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 0\right] + \left[\frac{1}{2}\left(-e^{-\infty}\right) - \frac{1}{2}\left(-e^{-1}\right)\right] \\
&= \left[\frac{1}{2} - 0\right] + \left[\frac{1}{2}\cdot(-0) - \frac{1}{2}\left(-\frac{1}{e}\right)\right] \\
&= \frac{1}{2} + \left[0 + \frac{1}{2e}\right] = \frac{1}{2} + \frac{1}{2e} \approx 0.6839 \neq 1\,,
\end{aligned}$$

so $g(x)$ is *not* a valid probability density. □

NOTE: The question originally had $g(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 0 \\ \frac{1}{2}e^{-x} & x \geq 0 \\ 0 & x < -1 \end{cases}$ instead, which would have made it a valid probability density. Given the phrasing of part **a** and the existence of part **b**, the change turned out to be pretty mean, confusing many of the students who tried to do this question. Sorry!

b. *(Solution i.)* Since $g(x)$ is *not* a valid probability density, $E(X)$ and $V(X)$ are meaningless, so there is nothing further to do ... □

b. *(Solution ii.)* Assuming that $g(x)$ was actually a valid probability density [whether because the solution to **a** was actually messed up, or was just assumed to be messed up], we compute $E(X)$ and $V(X)$ using their definitions and some calculus. First,

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xg(x)\,dx = \int_{-\infty}^{0} x\cdot 0\,dx + \int_0^1 x\cdot\frac{1}{2}\,dx + \int_1^{\infty} x\cdot\frac{1}{2}e^{-x}\,dx \\
&= 0 + \frac{1}{2}\cdot\frac{x^2}{2}\Big|_0^1 + \frac{1}{2}\left[-xe^{-x} + e^{-x}\right]\Big|_1^{\infty} \qquad \text{Using integration by parts with } u = x \text{ and } v' = e^{-x} \text{ to do the last integral.} \\
&= \frac{1^2}{4} - \frac{0^2}{4} + \frac{1}{2}\left[-\infty\cdot e^{-\infty} + e^{-\infty}\right] - \frac{1}{2}\left[-1\cdot e^{-1} + e^{-1}\right] \\
&= \frac{1}{4} - 0 + \frac{1}{2}[0+0] - \frac{1}{2}\left[-e^{-1} + e^{-1}\right] = \frac{1}{4} - 0 + \frac{1}{2}\cdot 0 - \frac{1}{2}\cdot 0 = \frac{1}{4}\,.
\end{aligned}$$

Second, with the help of some heavy recycling of parts of the calculation for $E(X)$,

$$\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 g(x)\,dx = \int_{-\infty}^{0} x^2 \cdot 0\,dx + \int_0^1 x^2 \cdot \frac{1}{2}\,dx + \int_1^\infty x^2 \cdot \frac{1}{2} e^{-x}\,dx \\
&= 0 + \frac{1}{2} \cdot \left.\frac{x^3}{3}\right|_0^1 + \frac{1}{2}\left[\left. -x^2 e^{-x} \right|_1^\infty - \int_1^\infty -2xe^{-x}\,dx \right] \quad \begin{array}{l}\text{Using parts with} \\ u = x^2 \text{ and } v' = e^{-x}.\end{array} \\
&= \frac{1^3}{6} - \frac{0^3}{6} + \frac{1}{2}\left[\left(-\infty^2 e^{-\infty}\right) - \left(-1^2 e^{-1}\right) + 2\left. \left[-xe^{-x} + e^{-x}\right] \right|_1^\infty \right] \\
&= \frac{1}{6} + \frac{1}{2}\left[0 + e^{-1} + 2 \cdot 0\right] = \frac{1}{6} + \frac{1}{2e}.
\end{aligned}$$

It follows that $V(X) = E\left(X^2\right) - [E(X)]^2 = \frac{1}{6} + \frac{1}{2e} - \left[\frac{1}{4}\right]^2 = \frac{1}{6} - \frac{1}{2e} - \frac{1}{16} = \frac{5}{48} - \frac{1}{2e}$. ■

7. A jar contains two identical red balls and three identical blue balls. Balls are drawn randomly from the jar, one at a time and without replacement, until all five balls have been removed from the jar.

- **a.** Find the sample space and probability function for this experiment. *[6]*
- **b.** Let A be the event that the third ball drawn is blue, and let B be the event that the first ball drawn is red. Determine whether A and B are independent. *[5]*
- **c.** The random variable R returns in which draw the second red ball came up. Compute the expected value $E(R)$ and variance $V(R)$ of R *[5]*

SOLUTIONS. **a.** $\Omega = \{\, BBBRR, BBRBR, BBRRB, BRBBR, BRBRB, BRRBB, RBBBR, RBBRB, RBRBB, RRBBB \,\}$. Each of these ten sequences of draws is equally likely, so the probability function is uniform, with $m(\omega) = \frac{1}{10}$ for each $\omega \in \Omega$. □

b. Since $A = \{\, BBBRR, BRBBR, BRBRB, RBBBR, RBBRB, RRBBB \,\}$ and $B = \{\, RBBBR, RBBRB, RBRBB, RRBBB \,\}$, $A \cap B = \{\, RBBBR, RBBRB, RRBBB \,\}$. Each outcome is equally likely and has a probability of $\frac{1}{10}$, so $P(A) = \frac{6}{10}$, $P(B) = \frac{4}{10}$, and $PA \cap B) = \frac{3}{10}$. As $P(A \cap B) = \frac{3}{10} = 0.3 \neq 0.24 = \frac{24}{100} = \frac{6}{10} \cdot \frac{4}{10} = P(A)P(B)$, the events A and B are not independent. □

c. The possible values of R are 2, 3, 4, and 5 – the *second* red ball can't very well turn up on the *first* draw. By counting how many outcomes in Ω give each value, we see that $P(R = 2) = \frac{1}{10}$, $P(R = 3) = \frac{2}{10}$, $P(R = 4) = \frac{3}{10}$, and $P(R = 5) = \frac{4}{10}$. We can now apply the definitions of $E(R)$ and $V(R)$:

$$E(R) = \sum_{k=2}^{5} kP(R = k) = 2 \cdot \frac{1}{10} + 3 \cdot \frac{2}{10} + 4 \cdot \frac{3}{10} + 5 \cdot \frac{4}{10} = \frac{2 + 6 + 12 + 20}{10} = \frac{40}{10} = 4$$

$$\begin{aligned}
E\left(R^2\right) &= \sum_{k=2}^{5} k^2 P(R = k) = 2^2 \cdot \frac{1}{10} + 3^2 \cdot \frac{2}{10} + 4^2 \cdot \frac{3}{10} + 5^2 \cdot \frac{4}{10} = \frac{4 + 18 + 48 + 100}{10} \\
&= \frac{170}{10} = 17
\end{aligned}$$

$$V(R) = E\left(R^2\right) - [E(R)]^2 = 17 - 4^2 = 17 - 16 = 1 \quad ■$$

8. Suppose X_1 and X_2 are independent continuous random variables that each have a standard normal distribution. Let $X = X_1 + X_2$.

a. Compute the expected value, $E(X)$, and variance, $V(X)$, of X. *[6]*

b. What is the distribution of X? *[10 = 4 if you guess right + 6 if you show it]*

SOLUTIONS. **a.** A standard normal distribution has expected value $\mu = 0$ and variance $\sigma^2 = 1^2 = 1$, so $E(X) = E(X_1 + X_2) = E(X_1) + E(X_2) = 0 + 0 = 0$ (by the linearity of E) and $V(X) = V(X_1 + X_2) = V(X_1) + V(X_2) = 1 + 1 = 2$ (since X_1 and X_2 are independent). $\square$

b. X has a normal distribution with expected value $\mu = 0$ and variance $\sigma^2 = 2$. (Hence the standard deviation of X is $\sigma = \sqrt{2}$.) This can be verified by working out the density function of X using the appropriate convolution integral. Recall that the density function for the standard normal distribution, and hence of X_1 and X_2, is $\varphi(t) = \dfrac{1}{\sqrt{2\pi}} e^{-t^2/2}$. Plugging this into the convolution formula gives:

$$
\begin{aligned}
f(x) = (\varphi * \varphi)(x) &= \int_{-\infty}^{\infty} \varphi(x-t)\varphi(t)\,dt = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} e^{-\left[(x-t)^2+t^2\right]/2}\,dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2\left[(t-x/2)^2+x^2/4\right]/2}\,dt \\
&\qquad \text{(Since } (x-t)^2 + t^2 = 2t^2 - 2xt + x^2 = 2\left[t^2 - xt + \tfrac{x^2}{4} - \tfrac{x^2}{4} + \tfrac{x^2}{2}\right] \\
&\qquad\qquad = 2\left[\left(\left(t - \tfrac{x}{2}\right)^2 + \tfrac{x^2}{4}\right]. \text{)} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(t-x/2)^2} e^{-x^2/4}\,dt = \frac{1}{2\pi} e^{-x^2/4} \int_{-\infty}^{\infty} e^{-(t-x/2)^2}\,dt \\
&\qquad \text{Now substitute } w = \sqrt{2}\left(t - \frac{x}{2}\right), \text{ so } dw = \sqrt{2}\,dt \text{ and } dt = \frac{1}{\sqrt{2}}\,dw. \\
&= \frac{1}{2\pi} e^{-x^2/4} \int_{-\infty}^{\infty} e^{-w^2/2} \cdot \frac{1}{\sqrt{2}}\,dw = \frac{1}{2\sqrt{\pi}} e^{-x^2/4} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-w^2/2}\,dw \\
&= \frac{1}{\sqrt{2} \cdot \sqrt{2\pi}} e^{-x^2/2\cdot\left(\sqrt{2}\right)^2} \qquad \text{(Since } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-w^2/2}\,dw = \int_{-\infty}^{\infty} \varphi(w)\,dw = 1.\text{)}
\end{aligned}
$$

That is, the density of X is $f(x) = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} e^{-x^2/2\cdot\left(\sqrt{2}\right)^2} = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} e^{-(x-0)^2/2\cdot\left(\sqrt{2}\right)^2}$, which is the density of a normal distribution with expected value $\mu = 0$ and variance $\sigma^2 = \left(\sqrt{2}\right)^2 = 2$, as required. $\blacksquare$

9. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	1	2	3
-1	0.1	0.1	0.2
1	0.1	0.2	0
3	0.2	0	0

a. Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\mathrm{Cov}(X, Y)$ of X and Y. *[12]*

b. Let $W = 2Y - 3X$. Compute $E(W)$ and $V(W)$. *[4]*

NOTE. The entries in the table only add up to 0.9 instead of 1, thanks to a typo. The entry for $X = 3$ and $Y = 3$ was supposed to have been 0.1 instead of 0. Sigh – this error escaped a weekend of proof-reading.

SOLUTIONS. **a.** Happily ignoring the noted problem, we compute away:

$$\begin{aligned}
E(X) &= (-1)(0.1 + 0.1 + 0.2) + 1(0.1 + 0.2 + 0) + 3(0.2 + 0 + 0) \\
&= (-1) \cdot 0.4 + 1 \cdot 0.3 + 3 \cdot 0.2 = -0.4 + 0.3 + 0.6 = 0.5 \\
E(Y) &= 1(0.1 + 0.1 + 0.2) + 2(0.1 + 0.2 + 0) + 3(0.2 + 0 + 0) \\
&= 1 \cdot 0.4 + 2 \cdot 0.3 + 3 \cdot 0.2 = 0.4 + 0.6 + 0.6 = 1.6 \\
E\left(X^2\right) &= (-1)^2(0.1 + 0.1 + 0.2) + 1^2(0.1 + 0.2 + 0) + 3^2(0.2 + 0 + 0) \\
&= 1 \cdot 0.4 + 1 \cdot 0.3 + 9 \cdot 0.2 = 0.4 + 0.3 + 1.8 = 2.5 \\
E\left(Y^2\right) &= 1^2(0.1 + 0.1 + 0.2) + 2^2(0.1 + 0.2 + 0) + 3^2(0.2 + 0 + 0) \\
&= 1 \cdot 0.4 + 4 \cdot 0.3 + 9 \cdot 0.2 = 0.4 + 1.2 + 1.8 = 3.4 \\
V(X) &= E\left(X^2\right) - [E(X)]^2 = 2.5 - 0.5^2 = 2.5 - 0.25 = 2.25 \\
V(Y) &= E\left(Y^2\right) - [E(Y)]^2 = 3.4 - 1.6^2 = 3.4 - 2.56 = 0.84 \\
E(XY) &= (-1) \cdot 1 \cdot 0.1 + (-1) \cdot 2 \cdot 0.1 + (-1) \cdot 3 \cdot 0.2 \\
&\quad + 1 \cdot 1 \cdot 0.1 + 1 \cdot 2 \cdot 0.2 + 1 \cdot 3 \cdot 0 \\
&\quad + 3 \cdot 1 \cdot 0.2 + 3 \cdot 2 \cdot 0 + 3 \cdot 3 \cdot 0 \\
&= -0.1 - 0.2 - 0.6 + 0.1 + 0.4 + 0 + 0.6 + 0 + 0 = 0.2 \\
\mathrm{Cov}(X, Y) &= E(XY) - E(X) \cdot E(Y) = 0.2 - 0.5 \cdot 1.6 = 0.2 - 0.8 = -0.6
\end{aligned}$$

Whew! □

b. Here hoes:

$$\begin{aligned}
E(W) &= E(2Y - 3X) = E(2Y) - E(3X) = 2E(Y) - 3E(X) \\
&= 2 \cdot 1.6 - 3 \cdot 0.5 = 3.2 - 1.5 = 1.7 \\
V(W) &= V(2Y - 3X) = V\left(2Y + (-3)X\right) = V(2Y) + V(-3X) + 2\mathrm{Cov}(2Y, -3X) \\
&= 2^2 V(Y) + (-3)^2 V(X) + 2 \cdot 2 \cdot (-3) \cdot \mathrm{Cov}(Y, X) \\
&= 4V(Y) + 9V(X) - 1.2 \cdot \mathrm{Cov}(X, Y) = 4 \cdot 0.84 + 9 \cdot 2.25 - 1.2 \cdot (-0.6) \\
&= 3.36 + 20.25 + 7.2 = 30.81 \qquad \blacksquare
\end{aligned}$$

[Total = 100]

Part Z. Bonus!

○•. Give as clever — *i.e.* with a simple set-up and an easy, but not obvious, solution — a probability problem as you can. *[1]*

•○. Write an original little poem about probability or mathematics in general. *[1]*

*[Part **X** is on page 1.]*

I HOPE THAT YOU ENJOYED THE COURSE. HAVE A GOOD THE SUMMER!