Trent University, Fall 2016

MATH 1550H Test

Monday, 22 February, 2016

Time: 50 minutes

Name: Solutions

Student Number: 0123456

Question	Mark	
1	____	
2	____	
3	____	
Total	____	/30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *three (3)* of **a**–**d**. *[12 = 3 × 4 each]*

a. The continuous random variable W has the density function $f(t) = \begin{cases} \frac{1}{4} & -2 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$.
Compute $P(W > 1)$.

b. How many distinct ways are there to arrange ten books, three of which are identical to one another, on four shelves? (Each shelf could accommodate all ten books.)

c. A five-card hand is drawn at random from a standard 52-card deck. What is the probability that each of the five cards is of a different kind?

d. A fair five-sided die with faces numbered 1 to 5 is rolled twice. Let X be the sum of the faces that come up on the two rolls. Find the probability function of X.

SOLUTIONS. **a.** Here goes:

$$\begin{aligned} P(W > 1) &= \int_1^\infty f(t)\,dt = \int_1^2 \frac{1}{4}\,dt + \int_2^\infty 0\,dt = \left.\frac{1}{4}t\right|_1^2 + 0 \\ &= \frac{1}{4}\cdot 2 - \frac{1}{4}\cdot 1 = \frac{2}{4} - \frac{1}{4} = \frac{1}{4} = 0.25 \qquad \square \end{aligned}$$

b. We will count the arrangements of ten books and three indistinguishable dividers: for each arrangement, the books before the first divider will be put on the first shelf in the order they occur in the arrangement, the books between the first and second dividers will be put on the second shelf in the order they occur in the arrangement, and so on. If the books and dividers were all distinct, there would be $(10 + 3)! = 13!$ ways to arrange them. Since the three dividers are indistiguishable, we have to divide this number by the number of ways one can arrange three distinct dividers, namely 3!; since three of the books are identical, we also have to divide by the number of ways one can arrange three distinct objects, namely 3!. It follows that there are $\dfrac{13!}{3!3!} = 172972800$ distinct ways to arrange ten books, three of which are identical to one another, on four shelves. $\square$

c. There are $\binom{13}{5}$ ways to pick the five kinds that appear in the hand and, since there are $\binom{4}{1} = 4$ ways to pick a card of each kind, there are $\binom{13}{5}4^5$ hands of five cards in which each card is of a different kind. Since there are $\binom{52}{5}$ equally likely hands, the probability that a randomly drawn hand of five cards has them all of different kinds is $\dfrac{\binom{13}{5}4^5}{\binom{52}{5}} \approx 0.5071$. $\square$

d. Note that since the die is fair, any particular pair of rolls (i, j) is as likely as any other; since the die is five sided, the probability of any particular pair is $\frac{1}{5}\cdot\frac{1}{5} = \frac{1}{25}$. Note that the least the sum could be is $1 + 1 = 2$ and the most it could be is $5 + 5 = 10$. We'll use this information and brute force to compute the probability function of X:

$$
\begin{aligned}
m(2) &= P(X=2) = P(1,1) = \frac{1}{25} \\
m(3) &= P(X=3) = P(1,2) + P(2,1) = \frac{2}{25} \\
m(4) &= P(X=4) = P(1,3) + P(2,2) + P(3,1) = \frac{3}{25} \\
m(5) &= P(X=5) = P(1,4) + P(2,3) + P(3,2) + P(4,1) = \frac{4}{25} \\
m(6) &= P(X=6) = P(1,5) + P(2,4) + P(3,3) + P(4,2) + P(5,1) = \frac{5}{25} \\
m(7) &= P(X=7) = P(2,5) + P(3,4) + P(4,3) + P(5,2) = \frac{4}{25} \\
m(8) &= P(X=8) = P(3,5) + P(4,4) + P(5,3) = \frac{3}{25} \\
m(9) &= P(X=9) = P(4,5) + P(5,4) = \frac{2}{25} \\
m(10) &= P(X=10) = P(5,5) = \frac{1}{25}
\end{aligned}
$$

... and $m(k) = 0$ otherwise. ■

2. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. If A and B are events in a sample space Ω, does $P(A|B) + P(A|\bar{B}) = P(A)$? Verify that it must be so or find an example demonstrating otherwise.

b. A fair coin is tossed five times. Let A be the event that exactly two heads occurred in the five tosses and B be the event that the first two tosses included one head and one tail. Determine whether A and B are independent or not.

c. The continuous random variable X has density function $g(t) = \begin{cases} 1 - \frac{1}{2}t & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$.

Find the *median* of X; that is, the number m such that $P(X \le m) = \frac{1}{2}$.

SOLUTIONS. **a.** $P(A|B)+P(A|\bar{B})$ need not equal $P(A)$. Suppose, for example, that $A = \Omega$ and B is any event such that $P(B) \neq 0$ and $P\left(\bar{B}\right) \neq 0$. Then we have:

$$P(A|B) + P\left(A|\bar{B}\right) = \frac{P(A \cap B)}{P(B)} + \frac{P\left(A \cap \bar{B}\right)}{P\left(\bar{B}\right)} = \frac{P(B)}{P(B)} + \frac{P\left(\bar{B}\right)}{P\left(\bar{B}\right)}$$
$$= 1 + 1 = 2 \neq 1 = P(\Omega) = P(A) \qquad \square$$

b. We need to check whether $P(A \cap B) = P(A)P(B)$, so we need to compute $P(A)$, $P(B)$, and $P(A \cap B)$.

Recall that A is the event that exactly two heads occurred in the five tosses. Since $P(H) = P(T) = \frac{1}{2}$ for any single toss of the coin, any particular sequence of five tosses has probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$. There are $\binom{5}{2} = 10$ choices for which two tosses out of five came up heads, so $P(A) = \binom{5}{2}\left(\frac{1}{2}\right)^5 = 10 \cdot \frac{1}{32} = \frac{5}{16}$.

B is the event that the first two tosses included one head and one tail, so $P(B) = P(HT) + P(TH) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Finally, $A \cap B$ is the event that the first two tosses included one head and one tail and the there were exactly two heads among the five tosses. That is, $A \cap B$ is the event that exactly one head occurred in the first two tosses and exactly one head occurred in the last three tosses. There are $\binom{2}{1} = 2$ ways to pick which of the first two tosses was the head, and $\binom{3}{1} = 3$ ways to pick which of the last three tosses has a head. As any particular sequence of five tosses has probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$, so $P(A \cap B) = \binom{2}{1}\binom{3}{1}\left(\frac{1}{2}\right)^5 = 2 \cdot 3 \cdot \frac{1}{32} = \frac{3}{16}$.

Thus $P(A \cap B) = \frac{3}{16} \neq \frac{5}{32} = \frac{5}{16} \cdot \frac{1}{2} = P(A)P(B)$, so A and B are not independent. $\square$

c. Given that the density function is 0 for $t < 0$ and for $t > 2$, there is no area under the graph except for $0 \le t \le 2$. It follows that the median m must be between 0 and 2. Thus

$$P(X \le m) = \int_{-\infty}^{m} f(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{m} \left(1 - \frac{1}{2}t\right)\,dt = 0 + \left(t - \frac{1}{2} \cdot \frac{t^2}{2}\right)\Big|_0^m$$
$$= \left(m - \frac{m^2}{4}\right) - \left(0 - \frac{0^2}{4}\right) = m - \frac{m^2}{4},$$

so the median must satisfy the equation $m - \frac{m^2}{4} = \frac{1}{2}$. Rearranging this to put everything on one side gives $\frac{m^2}{4} - m + \frac{1}{2} = 0$, and multiplying this by 4 to clear out the fractions gives us $m^2 - 4m + 2 = 0$. We can now apply the quadratic formula:

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

As noted above, m must be between 0 and 2; since $2 + \sqrt{2} > 2$ while $0 < 2 - \sqrt{2} < 2$, it follows that the median of X must be $m = 2 - \sqrt{2} \approx 0.5858$. ■

3. Do any *one (1)* of **a** or **b**. *[8 = 1 × 8 each]*

a. Supose the continuous random variable X has density function $g(t) = \begin{cases} e^{-t} & 0 \leq t \\ 0 & t < 0 \end{cases}$.
Let A be the event that $X > 2$ and B be the event that $X > 1$. Compute $P(A|B)$.

b. A hand of four cards is randomly chosen, without replacement, from a standard 52-card deck. What is the probability that at least one suit does not occur among the four cards?

SOLUTIONS. **a.** We need to compute $P(B)$ and $P(A \cap B)$. First, using the substitution $u = -t$, so $du = -dt$, from which it follows that $dt = (-1)\,du$, and $\begin{smallmatrix} t & 1 & \infty \\ u & -1 & -\infty \end{smallmatrix}$, we have:

$$\begin{aligned} P(B) = P(X > 1) &= \int_1^\infty g(t)\,dt = \int_1^\infty e^{-t}\,dt = \int_{-1}^{-\infty} e^u(-1)\,du \\ &= \int_{-\infty}^{-1} e^u\,du = e^u\big|_{-\infty}^{-1} = e^{-1} - e^{-\infty} = \frac{1}{e} - 0 = \frac{1}{e} \end{aligned}$$

Second, note that $A \cap B$ is the event that $X > 2$ and $X > 1$, which are both true exactly when $X > 2$, so $A \cap B = A$. Using he same substitution as before, with the limits changing as $\begin{smallmatrix} t & 2 & \infty \\ u & -2 & -\infty \end{smallmatrix}$, we have:

$$\begin{aligned} P(A \cap B) = P(A) = P(X > 2) &= \int_2^\infty g(t)\,dt = \int_2^\infty e^{-t}\,dt = \int_{-2}^{-\infty} e^u(-1)\,du \\ &= \int_{-\infty}^{-2} e^u\,du = e^u\big|_{-\infty}^{-2} = e^{-2} - e^{-\infty} = \frac{1}{e^2} - 0 = \frac{1}{e^2} \end{aligned}$$

It follows that $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/e^2}{1/e} = \frac{1}{e^2} \cdot \frac{e}{1} = \frac{1}{e} \approx 0.3679$. □

b. Note that there are $\binom{52}{4}$ hands of four cards that can be drawn from a standard 52-card deck, each equally likely to appear in a random draw.

We will use the inclusion-exclusion principle to count how many hands of four cards have at least one suit missing. There are $\binom{4}{1} = 4$ ways to choose a suit that will not occur in the hand and $\binom{39}{4}$ ways to choose a hand of four cards from the remaining cards. However, $\binom{4}{1}\binom{39}{4}$ overcounts the hands in which two or three suits are missing. Counting the hands with at least two suits missing, there are $\binom{4}{2} = 6$ ways to choose two suits that will not occur and $\binom{26}{4}$ ways to choose a hand of four cards from the remaining cards. However, $\binom{4}{2}\binom{26}{4}$ overcounts the hands with three suits missing, and hence $\binom{4}{1}\binom{39}{4} - \binom{4}{2}\binom{26}{4}$ undercounts the hands with three suits missing. Counting the hands with three suits missing (*i.e.* those in which only suit appears), there are $\binom{4}{3} = 4$ ways to choose three suits that will not occur and $\binom{13}{4}$ ways to choose a hand of four cards from the remaining cards. Thus there are $\binom{4}{1}\binom{39}{4} - \binom{4}{2}\binom{26}{4} + \binom{4}{3}\binom{13}{4}$ possible hands of four cards with at least one suit not occurring in the hand.

It follows that the probability of at least one suit not occurring in randomly drawn four-card hand is $\frac{\binom{4}{1}\binom{39}{4} - \binom{4}{2}\binom{26}{4} + \binom{4}{3}\binom{13}{4}}{\binom{52}{4}}$. [Simplify this yourself, if you dare!] ■

[Total = 30]