Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 2016

Solutions to Assignment #4
(Un)expected Value

1. Verify that $f(t) = \dfrac{1}{\pi\left(1+t^2\right)}$ is a probability density function, but that a random variable X that has $f(t)$ as its probability density does not have a finite expected value. *[7]*

Hint: Try computing $E(X)$ and see what you get ...

SOLUTION. First, we check that $f(t)$ is a valid probability density. $f(t) > 0$ for all t because $1+t^2$, 1, abd π are all positive for all values of t. $f(t)$ is also continuous, and hence integrable. It remains to show that the area under the entire graph is 1:

$$\int_{\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{\infty} \frac{1}{\pi\left(1+t^2\right)}\,dt = \frac{1}{\pi}\arctan(t)\Big|_{-\infty}^{\infty}$$
$$= \frac{1}{\pi}\arctan(\infty) - \frac{1}{\pi}\arctan(-\infty) = \frac{1}{\pi}\cdot\frac{\pi}{2} - \frac{1}{\pi}\cdot\left(-\frac{\pi}{2}\right) = \frac{1}{2}+\frac{1}{2} = 1$$

Thus $f(t)$ is indeed a probability density function. [I prefer to use $\arctan(t)$ instead of $\tan^{-1}(t)$ for the inverse function of the trig function $\tan(t)$, because people keep confusing $\tan^{-1}(t)$ with $\frac{1}{\tan(t)} = \cot(t)$.]

If a random variable X were to have $f(t)$ as its probability density, then its expected value would be:

$$E(X) = \int_{-\infty}^{\infty} t f(t)\,dt = \int_{-\infty}^{\infty} \frac{t}{\pi\left(1+t^2\right)}\,dt$$

Substitute $u = 1+t^2$, so $du = 2t\,dt$, and thus $t\,dt = \frac{1}{2}\,du$ and $\begin{array}{ccc} t & -\infty & \infty \\ u & \infty & \infty \end{array}$.

$$= \frac{1}{2\pi}\int_{\infty}^{\infty} \frac{1}{u}\,du = \frac{1}{2\pi}\ln(u)\Big|_{\infty}^{\infty} = \frac{1}{2\pi}\ln(\infty) - \frac{1}{2\pi}\ln(\infty) = \infty - \infty$$

Unfortunately, ∞ is *not* a real number and the expression $\infty - \infty$ has no clear definition, so $E(X)$ is not well-defined. [This is an over-simplification, but it's quicker than running through the defintion of improper integrals and using limits to the same end ...] $\square$

2. Find a function $g(t)$ such that a random variable X which has $g(t)$ as its probability density function has a finite expected value, but does not have a finite variance. *[3]*

SOLUTION. $g(t) = \begin{cases} 2t^{-3} & t \geq 1 \\ 0 & t < 1 \end{cases}$ is a simple example of such a probability density. I'll leave it to you to check that it is indeed a probability density. Now suppose a random variable X has $g(t)$ as its probability density. Then

$$E(X) = \int_{-\infty}^{\infty} t g(t)\,dt = \int_{-\infty}^{1} t\cdot 0\,dt + \int_{1}^{\infty} t\cdot 2t^{-3}\,dt = 0 + \int_{1}^{\infty} 2t^{-2}\,dt$$
$$= 2\frac{t^{-1}}{-1}\Big|_{1}^{\infty} = -\frac{2}{t}\Big|_{1}^{\infty} = \left(-\frac{2}{\infty}\right) - \left(-\frac{2}{1}\right) = 0-(-2) = 2\,,$$

so the expected value makes sense and is finite, but

$$
\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} t^2 g(t)\,dt = \int_{-\infty}^{1} t^2 \cdot 0\,dt + \int_{1}^{\infty} t^2 \cdot 2t^{-3}\,dt = 0 + \int_{1}^{\infty} 2t^{-1}\,dt \\
&= 2\ln(t)|_1^{\infty} = 2\ln(\infty) - 2\ln(1) = 2 \cdot \infty - 2 \cdot 0 = \infty\,,
\end{aligned}
$$

which is not finite, so neither is $V(X) = E\left(X^2\right) - [E(X)]^2$. ■