Mathematics 1550H – Introduction to probability
Trent University, Winter 2016

Solutions to Assignment #3
Irrational Bias

You die†. For your transgressions while still alive* you are initially placed by yourself in a featureless room with just one object: a thick biased coin which has a probability, when tossed, of getting a head of $P(H) = \frac{1}{\pi}$, a probability of getting a tail of $P(T) = \frac{1}{\sqrt{3}}$, and a probability of landing on edge of $P(E) = 1 - \frac{1}{\pi} - \frac{1}{\sqrt{3}}$. The Highest Authority gives you the following problems to solve‡, with the promise that if and when you solve them, you can move on to the rest of your afterlife.

1. How could you simulate a fair coin using the biased coin you have been given? *[2]*

Solution. Flip the biased coin twice. Let the outcome HT correspond to a head for the fair coin being simulated, the outcome TH correspond to a tail for the fair coin, and with any other outcome we simply repeat the process until we get an HT or a TH. This works for two reasons:

First, since individual tosses of the biased coin are independent $P(HT) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\pi} = P(TH)$, so the simulation is as likely to deliver a head of the fair coin as it is a tail. This means that the simulated coin is indeed fair.

Second, with probability 1, the process eventually delivers an HT or a TH. (Otherwise the simulation would run forever at least some of the time, which would be useless ... :-) Note that on any given pair of tosses of the biased coin, we have a probability of $P(HT) + P(TH) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\pi} = \frac{2}{\pi\sqrt{3}} \approx 0.3676$ of getting one of the desired outcomes and ending the simulation, and hence a probability of $1 - \frac{2}{\pi\sqrt{3}} \approx 0.6324$ of having to repeat the tosses. The probability of having to repeat the process forever is therefore $\left(1 - \frac{2}{\pi\sqrt{3}}\right)\left(1 - \frac{2}{\pi\sqrt{3}}\right)\left(1 - \frac{2}{\pi\sqrt{3}}\right) \cdots = \lim_{n\to\infty} \left(1 - \frac{2}{\pi\sqrt{3}}\right)^n = 0$. (Recall from calculus that $\lim_{n\to\infty} r^n = 0$ for any real number r with $|r| < 1$.) This means that the probability that the process will run forever and so not deliver a head or tail of the simulated coin is 0. ■

2. How could you simulate a fair standard six-sided die using the given coin? *[2]*

Solution. Flip the biased coin four times. Let the outcome $HHTT$ correspond to face 1 of the simulated die, $HTHT$ correspond to face 2, $HTTH$ correspond to face 3, $THHT$ correspond to face 4, $THTH$ correspond to face 5, and $TTHH$ correspond to face 6. If any other outcome occurs we simply repeat the process until we get one of the six desired outcomes.

This works because the desired outcomes are equally likely and the process terminates eventually with probability 1 by arguments similar to those in the solution to **1** above. ■

† Did you divide by zero? Nooooooo ...

* If you don't have worthy transgressions, like dividing by zero, just imagine that you did.

‡ No one expects to meet the Mathematical Inquisition once they're dead!

3. How could you simulate a biased coin with $P(H) = \frac{3}{5} = 0.6$ and $P(T) = \frac{2}{5} = 0.4$ using the given coin? *[2]*

Solution. Flip the given biased coin four times. Let the outcomes $HHTT$, $HTHT$, and $HTTH$ indicate a head of the simulated biased coin, and let the outcomes $THHT$ and $THTH$ indicate a tail of the simulated coin. If any other outcome occurs we repeat the process until we get one of the desired outcomes.

This works because the five outcomes are equally likely, by an argument like that in the solution to **1**, so the simulated coin will come up heads with a probability of $\frac{3}{5}$ when one of the five outcomes occurs, and come up tails with a probability of $\frac{2}{5}$ when one of the five outcomes occurs. (Note that the probabilities of the simulated coin are technically conditional probabilities of the process described above.) Moreover, the process will, with probability 1, eventually terminate, also by an argument like that in the solution to **1**. ■

4. How could you simulate a biased coin with $P(H) = \frac{1}{\sqrt{2}}$ and $P(T) = 1 - \frac{1}{\sqrt{2}}$ using the given coin? *[4]*

Note: $\frac{1}{\pi}$, $\frac{1}{\sqrt{3}}$, and $1 - \frac{1}{\pi} - \frac{1}{\sqrt{3}}$, as well as $\frac{1}{\sqrt{2}}$ and $1 - \frac{1}{\sqrt{2}}$, are all irrational, and so cannot be expressed as ratios of integers. Also, their decimal expansions (and expansions in other bases) are infinite and non-repeating.

Solution. To keep the process described below a little simpler, we'll assume that we are going to use a fair coin to simulate the biased coin with $P(H) = \frac{1}{\sqrt{2}}$ and $P(T) = 1 - \frac{1}{\sqrt{2}}$. This will be enough to solve the problem because we know from question **1** that we can use the given coin to simulate a fair coin. (Piling simulation on top of simulation is pretty wasteful, but the question did not ask for an *efficient* process ... :-)

We will need an expansion of $\frac{1}{\sqrt{2}}$ in base 2. This is like a base 10 (*i.e.* decimal) expansion, but done in terms of powers of 2 using the digits 0 and 1 instead of powers of 10 and the digits 0, 1, 2, ... , 9. Just as we have, in base 10,

$$\frac{1}{\sqrt{2}} = 0.70710678\cdots = \frac{7}{10} + \frac{0}{10^2} + \frac{7}{10^3} + \frac{1}{10^4} + \frac{0}{10^5} + \frac{6}{10^6} + \frac{7}{10^7} + \frac{8}{10^8} + \cdots,$$

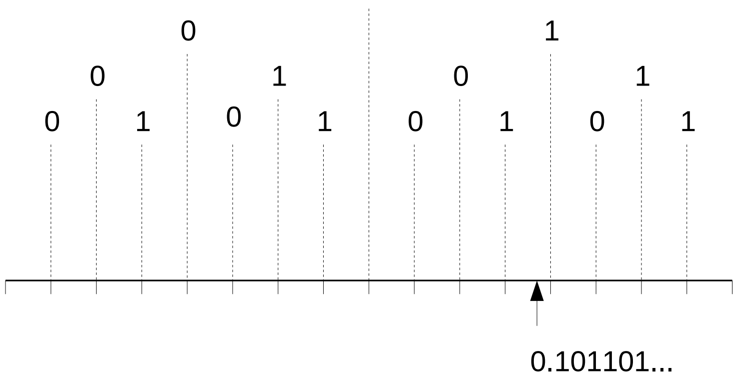
in base 2 we have

$$\frac{1}{\sqrt{2}} = 0.10110101\cdots = \frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \frac{0}{2^7} + \frac{1}{2^8} + \cdots$$

Since $\frac{1}{\sqrt{2}}$ is irrational, both expansions are infinite and non-repeating. We will assume that we have the entire base 2 expansion of $\frac{1}{\sqrt{2}}$. (More practically, one can compute as much of it as one needs, given sufficient time and tedium.)

The simulation process works as follows. Assign the digit 1 to the face H of the fair coin and the digit 0 to the face T. Toss the fair coin until first time the sequence of 0s and 1s it gives differs from the base 2 expansion of $\frac{1}{\sqrt{2}}$. If the differing digit given by the coin is a 0 and the corresponding digit of the base 2 expansion of $\frac{1}{\sqrt{2}}$ is 1, we have a head of the biased coin being simulated; if the differing digit given by the coin is a 1 and the corresponding digit of the base 2 expansion of $\frac{1}{\sqrt{2}}$ is 0, we have a tail of the biased coin being simulated.

Why does this do the job? If we tossed the fair coin infinitely many times, it would give the base 2 expansion of some real number in the interval $[0, 1]$. The first digit of the expansion would tell us if it's in the left (if the digit is 0) or right (if the digit is 1) half of the interval $[0, 1]$, the second digit tells us if it's in the left or right half of whichever half the first digit put it into, the third digit tells us if it's in the left or right half of that half of a half, and so on. Since the coin is fair, the number so generated is therefore as likely to be in any location in the interval $[0, 1]$ as it is in any other. It therefore has a probability of ending up in the subinterval $\left[0, \frac{1}{\sqrt{2}}\right]$, as opposed to ending up in the subinterval $\left[\frac{1}{\sqrt{2}}, 0\right]$, that is proportional to the length of these subintervals. That is, the number has a probability of $\frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$ of ending up in $\left[0, \frac{1}{\sqrt{2}}\right]$, and a probability of $1 - \frac{1}{\sqrt{2}}$ of ending up in $\left[\frac{1}{\sqrt{2}}, 0\right]$.



Our simulation process generates just enough of this random real number to determine which subinterval it ends up in.

The probability that the random real generated by tossing the fair coin will be (the base 2 expansion) of $\frac{1}{\sqrt{2}}$ (*i.e.* we'll be tossing the fair coin forever) is 0, by an argument similar to that used in the solution to **1**. ■