Mathematics 1550H – Introduction to probability
Trent University, Winter 2016

Solutions to Assignment #2
Hands across the deck

Recall that a standard 52-card deck has four suits, namely $\heartsuit$, $\diamondsuit$, $\clubsuit$, $\spadesuit$, each of which has thirteen cards, namely A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2. We will be dealing with five-card hands drawn at random, all at once, from the deck.

1. What is the probability that any particular five-card hand will be drawn? *[1]*

Solution. There are $\binom{52}{5} = \frac{52!}{(52-5)!5!} = 2598960$ five-card hands that can be drawn from the deck without replacement or order, each as likely as any other. Thus the probability that any particular five-card hand will be drawn is $\frac{1}{\binom{52}{5}} = \frac{1}{2598960} \approx 0.0000003848$. $\square$

Suppose that after a five-card hand is drawn, the cards in it are put back in the deck and another five-card hand is drawn.

2. What is the probability that the two hands have no card in common? *[1]*

Solution. If we avoid the five cards that were in the first hand, there are 47 cards left to choose the second hand from. It follows that there are $\binom{47}{5} = \frac{47!}{(47-5)!5!} = 1533939$ possible hands that have no card in common with the first hand. Since each particular hand has a probability of $\frac{1}{\binom{52}{5}}$ of being drawn (by the solution to **1** above), the probability that the second hand drawn has no card in common with the first is $\frac{\binom{52}{5}}{\binom{47}{5}} = \frac{\frac{52!}{47!5!}}{\frac{47!}{42!5!}} = \frac{1533939}{2598960} \approx 0.5902$. $\square$

3. What is the probability that the two hands have exactly one card in common? *[1]*

Solution. There are $\binom{52}{1} = 52$ ways to choose the common card, $\binom{51}{4} = \frac{51!}{47!4!} = 249900$ ways to choose the remaining four cards in the first hand, and $\binom{47}{4} = \frac{47!}{43!4!} = 178365$ ways to choose the remaining four cards in the second hand. Since there are $\binom{52}{5} = 2598960$ possible first hands, all equally likely, and $\binom{52}{5} = 2598960$ possible second hands, also all equally likely, the probability of the two hands having exactly one card in common is:

$$\begin{aligned} P(\text{exactly one card in common}) &= \frac{\binom{52}{1}\binom{51}{4}\binom{47}{4}}{\binom{52}{5}\binom{52}{5}} = \frac{52 \cdot 249900 \cdot 178365}{2598960 \cdot 2598960} \\ &\approx 0.3431 \qquad \square \end{aligned}$$

4. What is the probability that the two hands have at least one card in common? *[1]*

Solution. This is the complementary event to having no card in common, which probability was computed in the solution to **2**. Thus:

$$\begin{aligned} P(\text{at least one card in common}) &= 1 - P(\text{no card in common}) = 1 - \frac{1533939}{2598960} \\ &= \frac{1065021}{2598960} \approx 0.4098 \qquad \square \end{aligned}$$

5. What is the probability that the two hands have at least three cards in common? *[1]*

Solution. We'll take the direct approach – one could also use the answers to **2** and **3**, plus a calculation of the probability that the two hands had exactly two cards in common, to find the complementary probability. We will use the methods similar to those used in the solution to **3** to find the probabilities of the two hands having exactly three cards, exactly four cards, and exactly five cards, in common, respectively, and add these probabilities up.

First, there are $\binom{52}{3} = 22100$ ways to choose three common cards, $\binom{49}{2} = 1176$ ways to choose the remaining two cards in the first hand, and $\binom{47}{2} = 1081$ ways to choose the remaining two cards in the second hand. Since there are $\binom{52}{5} = 2598960$ possible first hands, all equally likely, and $\binom{52}{5} = 2598960$ possible second hands, also all equally likely, the probability of the two hands having exactly three cards in common is:

$$P(\text{exactly three cards in common}) = \frac{\binom{52}{3}\binom{49}{2}\binom{47}{2}}{\binom{52}{5}\binom{52}{5}} = \frac{22100 \cdot 1176 \cdot 1081}{2598960 \cdot 2598960} \approx 0.0042$$

Second, there are $\binom{52}{4} = 270725$ ways to choose three common cards, $\binom{48}{1} = 48$ ways to choose the remaining card in the first hand, and $\binom{47}{1} = 47$ ways to choose the remaining card in the second hand. Since there are $\binom{52}{5} = 2598960$ possible first hands, all equally likely, and $\binom{52}{5} = 2598960$ possible second hands, also all equally likely, the probability of the two hands having exactly four cards in common is:

$$P(\text{exactly four cards in common}) = \frac{\binom{52}{4}\binom{48}{1}\binom{47}{1}}{\binom{52}{5}\binom{52}{5}} = \frac{270725 \cdot 49 \cdot 47}{2598960 \cdot 2598960} \approx 0.0001$$

Third, there are $\binom{52}{5} = 2598960$ ways to choose the five common cards, and no need to choose any others. Since there are $\binom{52}{5} = 2598960$ possible first hands, all equally likely, and $\binom{52}{5} = 2598960$ possible second hands, also all equally likely, the probability of the two hands having exactly five cards in common is:

$$P(\text{exactly five cards in common}) = \frac{\binom{52}{5}}{\binom{52}{5}\binom{52}{5}} = \frac{1}{\binom{52}{5}} = \frac{1}{2598960} \approx 0.0000003848 \approx 0.0000$$

It follows that the probability that the two hands have at least three cards in common is:

$$\begin{aligned} P(\text{at least three cards in common}) &= P(\text{exactly three cards in common}) \\ &\quad + P(\text{exactly four cards in common}) \\ &\quad + P(\text{exactly five cards in common}) \\ &= \frac{\binom{52}{3}\binom{49}{2}\binom{47}{2}}{\binom{52}{5}\binom{52}{5}} + \frac{\binom{52}{4}\binom{48}{1}\binom{47}{1}}{\binom{52}{5}\binom{52}{5}} + \frac{1}{\binom{52}{5}} \\ &\approx 0.0042 + 0.0001 + 0.0000 = 0.0043 \qquad \square \end{aligned}$$

A *straight* is a hand in which the cards are in consecutive order by rank. For the purposes of the following question, going around the end of the rank order is not allowed. (For example, $4\,3\,2\,A\,K$ would not count as a straight.)

6. Suppose you draw a five-card hand randomly from the deck and get four cards that that would make a straight if you could replace the fifth card. (*e.g.* $J\,10\,9\,8\,3$ or $K\,7\,6\,4\,3$). If you are allowed to discard the fifth card and draw one at random from the remaining 47 cards in the deck, what is the probability that your modified hand will be a straight? *[3]*

Hint: There are several cases to consider ...

Solution. Note that if going around the end of the rank order is not allowed, and ignoring the suits of the cards, a straight looks like $A\ K\ Q\ J\ 10$, $K\ Q\ J\ 10\ 9$, $Q\ J\ 10\ 9\ 8$, $J\ 10\ 9\ 8\ 7$, 10 9 8 7 6, 9 8 7 6 5, 8 7 6 5 4, 7 6 5 4 3, or 6 5 4 3 2. It follows that a set of four cards which needs just one more to be a straight looks like one of:

i. Any one of the nine sets of five consecutive cards above with one of the middle three missing.
ii. $A\ K\ Q\ J$ or 5 4 3 2.
iii. Four consecutive kinds from $K\ Q\ J\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3$.

Observe that in cases *i* and *ii*, a straight can be completed only by filling one position, while in case *iii* one can complete a straight by adding a card at either end.

Suppose a hand of five cards contains an instance of cases *i* or *ii* and one other card. (The other card, given how the problem is phrased, is presumably *not* one that completes a straight.) There are four cards, one for each suit, that will complete the straight among the $52-5=47$ cards remaining in the deck. It follows that in cases *i* and *ii* the probability of completing the straight by discarding the other card and drawing one from the rest of the deck is $\frac{4}{47} \approx 0.0851$.

On the other hand, suppose a hand of five cards contains an instance of case *iii* and one other card. There are four cards, one for each suit, that will complete the straight at each end of the four consecutive cards, for a total of eight, among the $52-5=47$ cards remaining in the deck. It follows that in case *iii* the probability of completing the straight by discarding the other card and drawing one from the rest of the deck is $\frac{8}{47} \approx 0.1702$.

Sadly, this isn't quite the end of it if one wishes to know the overall probability of completing an almost straight by replacing the fifth card in the hand. We need to weight the probabilities in each case by the relative frequency of the case and then add them up.

As there are four of each kind in the deck, one for each suit, there are $4^4 = 256$ possible instances of each almost straight, of whatever case. Ignoring the fifth card in each hand, as it gets discarded anyway, we can see that the relative numbers of each case depend on the arrangement of kinds: there are nine sets of five cards times three choices of the middle one to miss, for a total of $9 \cdot 3 = 27$, for case *i*, two possibilities for case *ii*, and eight sequences of four consecutive kinds possible for case *iii*. There are therefore $27+2+8=37$ possible arrangements of kinds. It follows that the overall probability of completing an almost straight is given by:

$$P(\text{completion of almost straight}) = \frac{27}{37} \cdot \frac{4}{47} + \frac{2}{37} \cdot \frac{4}{47} + \frac{8}{37} \cdot \frac{8}{47} = \frac{180}{1739} \approx 0.1035 \qquad \square$$

Note. Should one really ignore the fifth card in the hand, as we did in computing the overall probability above? Consider, for example, a hand that looks like $K\ Q\ J\ 10\ 8$. This is an almost straight that is an instance of both case *i* and case *iii*. How does the analysis – and its outcome! – change when one takes such complications into account?

A *flush* is a hand in which all the cards are from the same suit.

7. Suppose you draw a five-card hand randomly from the deck. What is the probability that this hand is a flush? *[1]*

SOLUTION. There are $\binom{4}{1} = 4$ choices for the suit of the flush and $\binom{13}{5} = \frac{13!}{8!5!} = 1287$ ways to pick a five-card hand from that suit. Since there are $\binom{52}{5} = 2598960$ equally likely possible five-card hands, it follows that the probability of drawing a flush is $\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = \frac{4 \cdot 1287}{2598960} \approx 0.001981$. $\square$

8. Suppose you draw a five-card hand randomly from the deck. What is the probability that this hand is both a straight and a flush? *[1]*

SOLUTION. As noted in the solution to **6**, there are nine sequences of kinds that give a five-card straight: $A\ K\ Q\ J\ 10$, $K\ Q\ J\ 10\ 9$, ..., $6\ 5\ 4\ 3\ 2$. Thus there are nine straights that can be drawn from a single suit, and since there are four suits there are therefore $9 \cdot 4 = 36$ possible straights that are also flushes. It folows that the probability that a randomly drawn five-card hand is a straight flush is $\frac{36}{\binom{52}{5}} = \frac{36}{2598960} \approx 0.00001385$. $\blacksquare$