Mathematics 1550H – Introduction to probability
Trent University, Winter 2016

Partial Solutions to Assignment #1
Sheer drama or a comedy of errors?

A theatre with entirely reserved seating is sold out for a performance of Christopher Fry's *The Lady's Not For Burning,* and everyone who brought a ticket shows up. The theatre seats 128: the seats are duly numbered 1, 2, ... , 128, and each ticket has the number of the seat the member of the audience is to sit in. The audience enters the theatre one at a time. Unfortunately, the first one to enter doesn't notice the number of the seat on the ticket and sits in a random seat. After that, each person sits in their assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise.

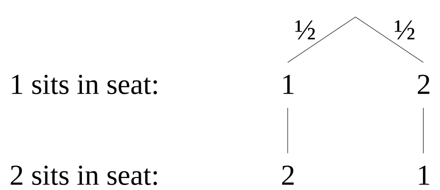
1. What is the probability that the last member of the audience to enter gets to sit in their assigned seat? *[10]*

The Short Solution. Consider the situation when the kth member of the audience enters. None of the previous ones showed any preference for the kth seat *vs.* the seat of the first to enter. This is, in particular, true when $k = n$. But the nth member of the audience can only occupy their own seat or the first one's seat. Therefore the probability is $\frac{1}{2} = 0.5$. ■

Note: This is basically the solution given in the compilation in which your instructor found the given problem†. It is a correct solution, though not really a complete one, because it leaves anyone reading it to work some key things out for themselves, such as why the last member of the audience can only end up in their own seat or the first one's seat. The phrase, common in some mathematical textbooks, that some detail(s) or proof(s) has been "left to the reader" should fill that reader with dread and terror ...

The Semi-Experimental Solution. We may as well assume that the audence comes in in order of ticket/seat number. (If not, simply have the house elves renumber the tickets and seats while they're entering to make it so. :-) Let's see what happens if, instead of having a theatre with 128 seats, we only have to deal with $n = 2$, 3, or 4.

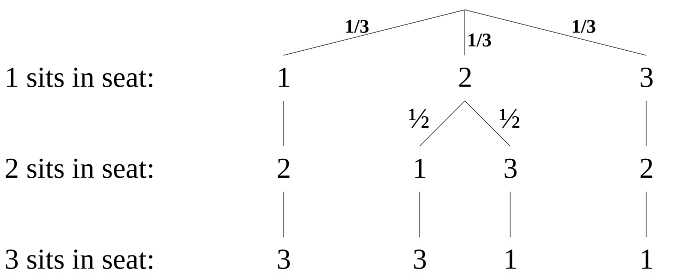
Here is the tree diagram for $n = 2$:



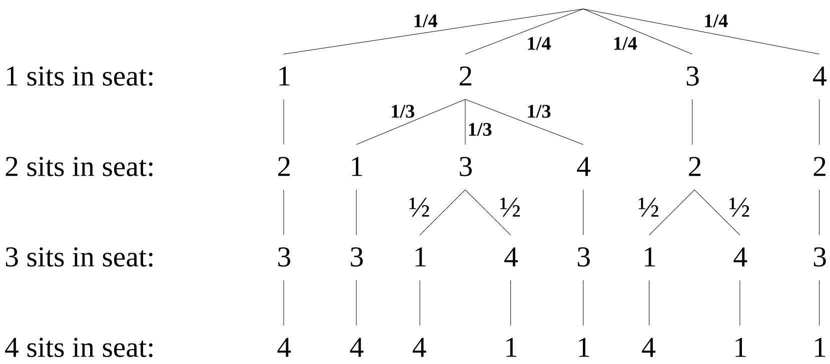
Note that after audience member 1 sits down, number 2 has no choice as to where to sit. Note also that the probability that the last member of the audience sits in the correct seat is indeed $\frac{1}{2} = 0.5$ in this case.

† The original version of this problem is due to P. Winkler.

The tree diagram for $n = 3$:



The tree diagram for $n = 4$:



Observe that in each case it is just as likely that the first member of the audience will sit in the correct seat as it is is that he or she will sit in the last one's seat, and that in both of these cases everyone in between occupies their correct seat. Moreover, if the first member sits in a in-between seat, then the tree below that point is essentially a copy of a tree from a case with fewer people. Since in these trees it is also just as likely that that the last member will get their seat as it is that they will not, the probability that the last member gets to sit in the correct seat is still $\frac{1}{2} = 0.5$.

These observations scale as the size of the audience, and hence the trees, get larger – grind out the trees for $n = 5$ and/or $n = 6$ yourself if you're not convinced – so even with $n = 128$ members of the audience, the probability that the last member gets to sit in the correct seat is $\frac{1}{2} = 0.5$. $\square$

Note: The problem with this solution is that it is not quite a proof: how do you really know that the pattern with the trees and probabilities really does scale as you increase the size of the audience?