Mathematics 1550H – Introduction to probability
TRENT UNIVERSITY, Winter 2016

Solutions to the Quizzes

NOTE: *Most quiz problems have more than one correct solution, so if your solution looks different from those given here, you are not necessarily wrong. (Besides, your instructor also makes mistakes, so these solutions aren't necessarily correct... :-)*

Quiz #1. Friday, 15 January, 2016. *[15 minutes]*

A fair four-sided die has its faces marked with the numbers 1, 2, 3, and 4, respectively. Since the die is fair, any of these numbers is as likely to be rolled as any other. The die is rolled twice.

1. What is the sample space Ω for this experiment? How many outcomes are there in Ω? *[2]*

Find the probability that

2. ... both rolls gave an odd number. *[1]*
3. ... at least one of the rolls gave an even number. *[1]*
4. ... one of the rolls gave an even number and the other gave an odd number? *[1]*

SOLUTIONS. 1. The sample space consists of all ordered pairs with each element of the pair being one of the numbers 1–4. Explicitly:

$$\begin{aligned}\Omega = \{\, &(1,1),\ (1,2),\ (1,3),\ (1,4),\ (2,1),\ (2,2),\ (2,3),\ (2,4),\\ &(3,1),\ (3,2),\ (3,3),\ (3,4),\ (4,1),\ (4,2),\ (4,3),\ (4,4)\,\}\end{aligned}$$

There are $4 \cdot 4 = 16$ outcomes in Ω. □

Note that because each outcome is equally likely, it follows that we have a uniform distribution with $m(\omega) = \frac{1}{16}$ for each $\omega \in \Omega$.

2. Let A be the event that both rolls gave an odd number. Then $A = \{\,(1,1),\ (1,3),\ (3,1),\ (3,3)\,\}$, so, by definition:

$$P(A) = m(1,1) + m(1,3) + m(3,1) + m(3,3) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4} = 0.25 \qquad \square$$

3. The event that at least one of the rolls gave an even number occurs exactly when it is not the case that both rolls gave an odd number, so it is the complement of the event A in the solution to question 2. Thus the probability that at least one of the rolls gave an even number is $P\left(\overline{A}\right) = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$. □

4. Let B be the event that one of the rolls gave an even number and the other gave an odd number. Then

$$B = \{\,(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\,\}\ ,$$

so, by definition:

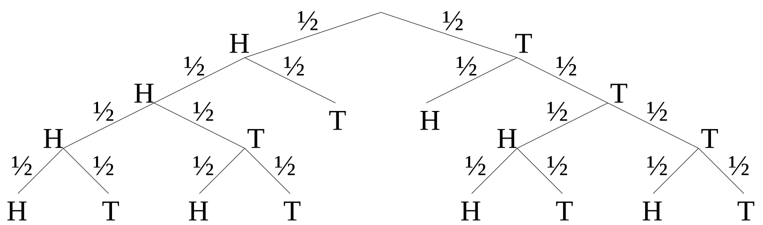
$$\begin{aligned}P(B) &= m(1,2) + m(1,4) + m(2,1) + m(2,3) + m(3,2) + m(3,4) + m(4,1) + m(4,3)\\ &= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{8}{16} = \frac{1}{2} = 0.5 \qquad \blacksquare\end{aligned}$$

Quiz #2. Friday, 22 January, 2016. *[12 minutes]*

A fair coin is tossed twice. If the two tosses came up with different faces, that's it; otherwise, if the two tosses came up with the same face, the coin is tossed two more times.

1. Draw the complete tree diagram for this experiment. *[1.5]*
2. What is the probability that at least two tosses in a row came up the same way during the experiment? *[1]*
3. What is the probability that exactly three tosses, not necessarily in a row, came up the same way during the experiment? *[1]*
4. What would the sample space and probability function be if the coin were tossed until five tosses had been made or the coin came up with whatever face appeared on the first toss again, whichever came first? *[1.5]*

SOLUTIONS. 1. Here is the complete tree diagram:



Note that there are ten outcomes, two of which have a probability of $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ each, and eight of which have a probability of $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$ each. □

2. Looking at the tree, the only way not to have at least two tosses in a row come up the same way in this process is to have different faces come up on the first two tosses. (In that case you stop without tossing further; otherwise, you already have two consecutive tosses that are the same.) Thus the probability of the event A that at least two tosses in a row came up the same way is given by $P(A) = 1 - P\left(\overline{A}\right) = 1 - [m(HT) + m(TH)] = 1 - \left[\frac{1}{4} + \frac{1}{4}\right] = 1 - \frac{1}{2} = \frac{1}{2} = 0.5$. □

3. Examining the tree, we see that the only outcomes with exactly three heads are $HHHT$ and $HHTH$, and the only outcomes with exactly three tails are $TTHT$ and $TTTH$. All four of these have a probability of $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$ each, so the probability of having exactly three tosses come up the same way in this experiment is $4 \cdot \frac{1}{16} = \frac{4}{16} = \frac{1}{4} = 0.25$. □

4. In this case the sample space is

$$\Omega = \{HH, TT, HTH, THT, HTTH, THHT, HTTTH, HTTTT, THHHT, THHHH\},$$

and the probability function is given by:

$$m(HH) = m(TT) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$m(HTH) = m(THT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$$m(HTTH) = m(THHT) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$$

$$m(HTTTH) = m(HTTTT) = m(THHHT) = m(THHHH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$$

The coin is fair, so the probability of any particular sequence of heads and/or tails is just $\left(\frac{1}{2}\right)^n$, where n is the elength of the sequence. ■

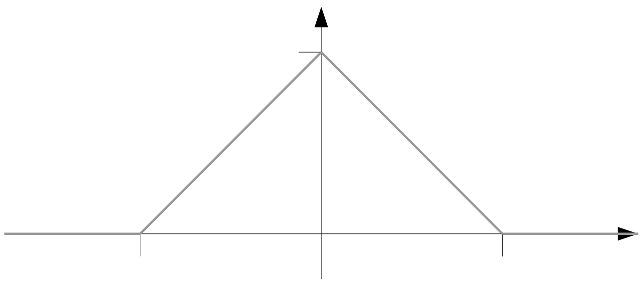
Quiz #3. Friday, 29 January, 2016. *[10 minutes]*

Suppose that the continuous random variable X has the probability density function:

$$f(t) = \begin{cases} 1+t & -1 \leq t \leq 0 \\ 1-t & 0 \leq t \leq 1 \\ 0 & t < -1 \text{ or } t > 1 \end{cases}$$

1. Graph $f(t)$. *[1]*
2. Verify that $f(t)$ is indeed a probability density function. *[2]*
3. Compute $P\left(X \geq \frac{1}{2}\right)$. *[2]*

SOLUTIONS. 1. Here is a crude sketch of the graph of $f(t)$:



□

2. *[Without calculus.]* We need to check three conditions. First, $f(t)$ is integrable because it is continuous. [Since every function we encounter in this course will be integrable, as was noted in class, no points will be taken off if you didn't check for integrability. Enjoy the freebie!] Second, it is obvious from the graph that $f(t) \geq 0$ for all t. Third, $\int_{-\infty}^{\infty} f(t)\,dt = 1$, *i.e.* the total area under the entire graph of $f(t)$ is 1, because only the part of $f(t)$ from -1 to 1 contributes any positive area, and that is a triangle with base $2 = 1-(-1)$ and height 1, which has area $\frac{1}{2} \cdot 2 \cdot 1 = 1$. Thus $f(t)$ does satisfy the conditions necessary to be a probability density. □

NOTE: Of course, one could also check that the area under the graph of $f(t)$ is 1 by actually working out $\int_{-\infty}^{\infty} f(t)\,dt$ using integral calculus.

3. *[With calculus.]* By definition, $P\left(X \geq \frac{1}{2}\right)$ is the area under the graph of $f(t)$ for $\frac{1}{2} \leq t$:

$$\begin{aligned} P\left(X \geq \frac{1}{2}\right) &= \int_{1/2}^{\infty} f(t)\,dt = \int_{1/2}^{1} (1-t)\,dt + \int_{1}^{\infty} 0\,dt \\ &= \left(t - \frac{t^2}{2}\right)\Big|_{1/2}^{1} + 0 = \left(1 - \frac{1^2}{2}\right) - \left(\frac{1}{2} - \frac{(1/2)^2}{2}\right) \\ &= \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{8}\right) = 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{8} = 1 - \frac{1}{8} = \frac{7}{8} = 0.875 \qquad \blacksquare \end{aligned}$$

NOTE: Of course, one could also compute the area under the graph of $f(t)$ for $\frac{1}{2} \leq t$ without calculus . . .

Quiz #4. Friday, 5 February, 2016. *[10 minutes]*

1. How many distinct ways are there to take all the letters, including the repetitions, in the word "Mississauga" and arrange them in a row? *[5]*

SOLUTION. "Mississauga" has eleven letters. If they were all distinct, there would be $11! = 11 \cdot 10 \cdot 9 \cdots 3 \cdot 2 \cdot 1$ ways to arrange them in a row. However, there are four copies of "s", and two each of "a" and "i", in any arrangement of the letters. Rearranging the four copies of "s", or the two copies of "a", or the two copies of "i", among themselves in the positions they occupy will not change the overall arrangement. (For example, if we swap adjacent copies of "s" with each other in the original word, it will look like "Mississauga". See the difference? :-) As there are 4! ways to arrange four objects and 2! ways to arrange two objects, respectively, it follows that there are

$$\frac{11!}{4!2!2!} = \frac{39916800}{24 \cdot 2 \cdot} = 415800$$

distinct ways to arrange all the letters in the word "Mississauga" . ■

Quiz #5. Friday, 12 February, 2016. *[12 minutes]*

A fair coin is tossed six times. Let A be the event that the number of heads that came up is odd, and let B be the event that the number of heads that came up is grreater than two.

1. Compute $P(A|B)$. *[5]*

SOLUTION. We need to determine $P(B)$ and $P(A \cap B)$. First, however, observe that the probability of obtaining exactly k heads in six tosses of a fair coin is $P(\text{exactly } k \text{ Hs}) = \binom{6}{k}\left(\frac{1}{2}\right)^6$ because there are $\binom{6}{k}$ ways to choose the k tosses in which the heads occur, and any particular sequence of six tosses of a fair coin has probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^6$. It follows that

$$\begin{aligned} P(B) &= 1 - P\left(\overline{B}\right) = 1 - P(\text{number of Hs is} \leq 2) \\ &= 1 - [P(\text{exactly 0 Hs}) + P(\text{exactly 1 H}) + P(\text{exactly 2 Hs})] \\ &= 1 - \left[\binom{6}{0}\left(\frac{1}{2}\right)^6 + \binom{6}{1}\left(\frac{1}{2}\right)^6 + \binom{6}{2}\left(\frac{1}{2}\right)^6\right] \\ &= 1 - \left[1 \cdot \frac{1}{64} + 6 \cdot \frac{1}{64} + 15 \cdot \frac{1}{64}\right] = 1 - \frac{22}{64} = \frac{42}{64} = \frac{21}{32} = 0.65625 \end{aligned}$$

and

$$\begin{aligned} P(A \cap B) &= P(\text{number of Hs is odd and} > 2) = P(\text{exactly 3 Hs}) + P(\text{exactly 5 Hs}) \\ &= \binom{6}{3}\left(\frac{1}{2}\right)^6 + \binom{6}{5}\left(\frac{1}{2}\right)^6 = 20 \cdot \frac{1}{64} + 6 \cdot \frac{1}{64} = \frac{26}{64} = \frac{13}{32} = 0.203125\,, \end{aligned}$$

so

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{13}{32}}{\frac{21}{32}} = \frac{13}{32} \cdot \frac{32}{21} = \frac{13}{21} \approx 0.619048\,. \qquad ■$$

Quiz #6. Friday, 4 March, 2016. *[15 minutes]*

1. Suppose the continuous random variable H has density function $h(t) = \begin{cases} t^{-2} & 1 \leq t \\ 0 & t < 1 \end{cases}$. Let A be the event that $H \leq 2$ and B be the event that $\frac{4}{3} \leq H \leq 4$. Determine whether A and B are independent. *[5]*

SOLUTION. Events A and B are independent if $P(A \cap B) = P(A)P(B)$, and dependent otherwise, so we need to compute $P(A)$, $P(B)$, and $P(A \cap B)$. To save time and space later on, note that the Power Rule for integration tells us that:

$$\int t^{-2}\,dt = \frac{t^{-2+1}}{-2+1} + C = \frac{t^{-1}}{-1} + C = -t^{-1} + C = -\frac{1}{t} + C$$

[Of course, the constant C cancels out, and hence is normally ignored, when we compute definite integrals.] With this in hand, here we go:

$$\begin{aligned}
P(A) &= P(H \leq 2) = \int_{-\infty}^{2} h(t)\,dt = \int_{-\infty}^{1} 0\,dt + \int_{1}^{2} t^{-2}\,dt \\
&= 0 + -\frac{1}{t}\bigg|_1^2 = \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right) = -\frac{1}{2} + 1 = \frac{1}{2} \\
P(B) &= P\left(\frac{4}{3} \leq H \leq 4\right) = \int_{4/3}^{4} h(t)\,dt = \int_{4/3}^{4} t^{-2}\,dt \\
&= -\frac{1}{t}\bigg|_{4/3}^{4} = \left(-\frac{1}{4}\right) - \left(-\frac{1}{4/3}\right) = -\frac{1}{4} + \frac{3}{4} = \frac{1}{2} \\
P(A \cap B) &= P\left(\frac{4}{3} \leq H \leq 2\right) = \int_{4/3}^{2} h(t)\,dt = \int_{4/3}^{2} t^{-2}\,dt \\
&= -\frac{1}{t}\bigg|_{4/3}^{2} = \left(-\frac{1}{2}\right) - \left(-\frac{1}{4/3}\right) = -\frac{1}{2} + \frac{3}{4} = \frac{1}{4}
\end{aligned}$$

Since $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A)P(B)$, the events A and B are independent. ■

Quiz #7. Friday, 11 March, 2016. *[10 minutes]*

1. Suppose a non-standard fair six-sided die has faces numbered 1, 3, 4, 5, 7, and 8, respectively. What is the expected value of the of the number that comes up if the die is rolled once? *[5]*

SOLUTION. Since the die is fair, any of the six faces is as likely to come up as any other, so each face has a probability of $\frac{1}{6}$ of coming up. If X is the number that comes that comes up when the die is rolled, then its expected value is:

$$\begin{aligned}
E(X) &= 1 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 7 \cdot \frac{1}{6} + 8 \cdot \frac{1}{6} \\
&= \frac{1+3+4+5+7+8}{6} = \frac{28}{6} = \frac{14}{3} \approx 4.6667 \qquad \blacksquare
\end{aligned}$$

Quiz #8. Friday, 18 March, 2016. *[10 minutes]*

A fair coin is tossed until the second head comes up.

1. What is the probability that at most four tosses are required? *[3]*
2. What are the expected value and variance of the number of tosses required? *[2]*

SOLUTION. 1. *[Brute force.]* If the coin is tossed until the second head comes up, the possible outcomes are HH, HTH, THH, $HTTH$, $THTH$, $TTHH$, $HTTTH$, ... Since the coin is fair, $P(H) = P(T) = \frac{1}{2}$ on any one toss. It follows that

$$\begin{aligned} P(\leq 4 \text{ tosses}) &= P(HH) + P(HTH) + P(THH) + P(HTTH) + P(THTH) + P(TTHH) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{15}{16} = 0.9375 \qquad \square \end{aligned}$$

1. *[Recognizing the distribution.]* The number of tosses required until a second head occurs has a negative binomial distribution with $k = 2$ and, since the coin is fair, $p = q = \frac{1}{2}$. This has probability function $m(x) = \binom{x-1}{k-1} p^k q^{x-k} = \binom{x-1}{k-1} \left(\frac{1}{2}\right)^x$ for $x = 2, 3, \ldots$ and $m(x) = 0$ otherwise. It follows that:

$$\begin{aligned} P(\leq 4 \text{ tosses}) &= P(2 \text{ tosses}) + P(3 \text{ tosses}) + P(4 \text{ tosses}) = m(2) + m(3) + m(4) \\ &= \binom{2-1}{2-1} \left(\frac{1}{2}\right)^2 + \binom{3-1}{2-1} \left(\frac{1}{2}\right)^3 + \binom{4-1}{2-1} \left(\frac{1}{2}\right)^4 \\ &= 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} = \frac{15}{16} = 0.9375 \qquad \square \end{aligned}$$

2. *[Recognizing the distribution.]* A random variable X that has a negative binomial distribution with $k = 2$ and $p = q = \frac{1}{2}$ has expected value $E(X) = \frac{k}{p} = \frac{2}{\frac{1}{2}} = 4$ and variance $V(X) = \frac{kq}{p^2} = \frac{2 \cdot \frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 4$. ■

NOTE. Trying to do question 2 by brute force requires a lot of work ...

Quiz #9. Friday, 1 April, 2016. *[10 minutes]*

The continuous random variable X has a normal distribution with expected value $\mu = 3$ and variance $\sigma^2 = 4$.

1. Use a table for the standard normal distribution to approximate $P(1 \leq X \leq 5)$. *[2.5]*
2. Use Chebyshev's Inequality to estimate $P(1 \leq X \leq 5)$. *[2.5]*

SOLUTION. 1. Let $Z = \frac{X-\mu}{\sigma} = \frac{X-3}{2}$ and recall that Z has a standard normal distribution. Since

$$1 \leq X \leq 6 \Leftrightarrow -2 = 1 - 3 \leq X - 3 \leq 5 - 3 = 2 \Leftrightarrow -1 = \frac{-2}{2} \leq Z = \frac{X-3}{2} \leq \frac{2}{2} = 1,$$

we have, consulting the standard normal table, that $P(1 \leq X \leq 6) = P(-1 \leq Z \leq 1) = P(Z \leq 1) - P(Z < -1) \approx 0.8413 - 0.1587 = 0.6286$. $\square$

2. Chebyshev's Inequality tells us that for any $t > 0$, $P(|X - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$. In this case, $1 \leq X \leq 6 \Leftrightarrow -2 = 1 - 3 \leq X - 3 \leq 5 - 3 = 2 \Leftrightarrow |X - 3| \leq 2$. Since Chebyshev's Inequality tells us that $P(|X - 3| \leq 2) \geq \frac{2^2}{2^2} = 1$, we can only conclude that $P(1 \leq X \leq 5) = P(|X - 3| \leq 2) = 1 - P(|X - 3| \geq 2) \geq 1 - 1 = 0$. (Not very useful!). ■