Mathematics 1550H – Introduction to probability

Trent University, Winter 2016

Final Examination

Monday, 11 April, 2016

Tempus-Locus: 14:00–17:00 in the Gym *Inflicted by* Стефан Біланюк.

Instructions: Do both of parts **P** and **Q**, and, if you wish, part **Ω**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; standard normal table; mid-ears space.

Part P. Do all of **1**–**5**. *[Subtotal = 68/100]*

1. A biased coin with $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$ is tossed three times.

a. Draw the complete tree diagram for this experiment. *[5]*

b. What are the sample space and probability function for this experiment? *[5]*

c. What is the probability that exactly one tail came up in this experiment? *[5]*

2. The continuous random variable Y has the following probability density function:

$$f(t) = \begin{cases} \frac{2}{3}t & 0 \le t \le 1 \\ 1 - \frac{1}{3}t & 1 \le t \le 3 \\ 0 & t < 0 \text{ or } t > 3 \end{cases}$$

a. Verify that $f(t)$ is indeed a probability density function. *[6]*

b. Compute $P(Y \le 2)$. *[4]*

3. A hand of five cards is drawn simultaneously (without order or replacement) from a standard 52-card deck. Let A be the event that the hand includes four cards of the same kind, and let B be the event that at least two of the cards in the hand are of the same kind.

a. Compute $P(A)$. *[5]* **b.** Compute $P(B)$. *[5]* **c.** Compute $P(A|B)$. *[3]*

4. Suppose Z and X are continuous random variables such that Z has a standard normal distribution and $X = 5Z + 10$.

a. Compute $P(7 \le X \le 17)$. *[6]*

b. What are the expected value $E(X)$ and variance $V(X)$ of X? *[6]*

c. What kind of distribution does X have? *[3]*

5. Suppose X is a discrete random variable that has a geometric distribution with $p = \frac{1}{2}$.

a. Compute $P(X \ge 6)$. *[5]*

b. Use Markov's Inequality to estimate $P(X \ge 6)$. *[5]*

c. Use Chebyshev's Inequality to estimate $P(X \ge 6)$. *[5]*

*[Parts **Q** and **Ω** are on page 2.]*

Part Q. Do any *two* (2) of **6–9**. *[Subtotal = 32/100]*

6. Suppose $h(t) = \begin{cases} e^t & t \leq 0 \\ 0 & t > 0 \end{cases}$ is the probability density function of the continuous random variable W.

- **a.** Verify that $h(t)$ is indeed a probability density function. *[8]*
- **b.** Compute the expected value $E(W)$ and variance $V(W)$ of W. *[8]*

7. Suppose that an experiment consists of tossing a fair coin until it comes up heads, and suppose the discrete random variables X_1, X_2, and X_3 count the number of tosses required in three separate and independent runs of this experiment. What is the probability distribution function of $X = X_1 + X_2 + X_3$? Why? *[16]*

8. A jar contains 6 white beads and 3 black beads. Beads are chosen randomly from the jar one at a time until the third time a black bead turns up.

- **a.** Suppose that each bead is replaced before the next is chosen. How many beads should you expect to be chosen in the course of the experiment? *[6]*
- **b.** Suppose that if a bead is white, it is *not* replaced before the next bead is chosen, but if it is black, it *is* replaced before the next is chosen. How many beads should you expect to be chosen in the course of the experiment? *[10]*

9. Suppose the discrete random variables U and W are jointly distributed according to the following table:

$U \backslash W$	1	2
2	0.1	0.3
3	0.2	0.1
4	0.1	0.2

- **a.** Compute the expected values $E(U)$ and $E(W)$, variances $V(U)$ and $V(W)$, and covariance $\mathrm{Cov}(U, W)$ of U and W. *[12]*
- **b.** Let $X = U + W$. Compute $E(X)$ and $V(X)$. *[4]*

[Total = 100]

Part Ω. Bonus!

••. In series of games numbered 1, 2, 3, ..., the winning number in the nth game is randomly chosen from the set $\{1, 2, \ldots, n+2\}$. Kosh Naranek bets on 1 in each game and intends to keep playing until winning once. What is the probability that Kosh will have to play forever? Why? *[1]*

∘∘. Write an original little poem about probability or mathematics in general. *[1]*

*[Part **P** is on page 1.]*

I HOPE THAT YOU ENJOYED THE COURSE.
ENJOY THE SUMMER!