Trent University, Winter 2015

MATH 1550H Test

Thursday, 26 February, 2015

Time: 50 minutes

Name: Solutions

Student Number: 0314159

Question	Mark	
1	____	
2	____	
3	____	
Total	____	/30

Instructions
- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator and an aid sheet.

1. Do any *three* (3) of **a**–**d**. *[12 = 3 × 4 each]*

a. Two cards are simultaneously drawn at random from a standard 52-card deck. Let E be the event that exactly one of the two cards is a $\diamondsuit$, and F be the event that exactly of the two one cards is a $\heartsuit$. Compute $P(E|F)$.

b. How many ways are there to arrange four books numbered 1–4 on a shelf so that book number 2 occurs somewhere to the right of book number 1?

c. A fair coin is tossed repeatedly. What is the probability that exactly two tails come up before the second head occurs?

d. A fair standard die is rolled twice. Compute the probability that the sum of the two rolls is an odd number.

SOLUTIONS. **a.** $P(E|F) = \frac{P(E\cap F)}{P(F)}$, so we need to compute $P(F)$ and $P(E \cap F)$. Note there are $\binom{52}{2} = \frac{52!}{50!2!} = \frac{52\cdot 51}{2} = 26 \cdot 51 = 1326$ ways to choose two cards from a deck of 52 cards, all equally likely.

There are $\binom{13}{1} = 13$ ways to choose one $\heartsuit$ and $\binom{39}{1} = 39$ ways to choose one card that is not a $\heartsuit$. It follows that there are $\binom{13}{1}\binom{39}{1} = 13 \cdot 39 = 507$ ways to choose two cards such that exactly one is a $\heartsuit$, and so $P(F) = \frac{507}{1326} \approx 0.3824$.

$E \cap F$ is the event that one of the two cards is a $\heartsuit$ and the other is a $\diamondsuit$. There are $\binom{13}{1} = 13$ ways to choose one $\diamondsuit$, so there are $\binom{13}{1}\binom{13}{1} = 13 \cdot 13 = 169$ ways to choose one $\heartsuit$ and one $\diamondsuit$. It follows that $P(E \cap F) = \frac{169}{1326} \approx 0.1275$.

Thus $P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{169/1326}{507/1326} = \frac{169}{507} = \frac{1}{3} \approx 0.3333$. □

NOTE: That was the straighforward way to do **a**, but there is a simpler way to get the answer with just a bit of insight ... *Hint:* How many non-$\heartsuit$ suites are there?

b. There are $4! = 24$ ways to arrange four distinct books in a row. Any arrangement that has book number 2 somewhere to the left of book number 1 can be converted into an arrangement that has book number 2 somewhere to the right of book number 1 by simply swapping the books, and *vice versa.* Since every arrangement has to have book number 2 on one side or the other of book number 1, it follows that there are just as many arrangments that have book number 2 somewhere to the right of book number 1 as arrangements that do not. Thus there are $\frac{24}{2} = 12$ arrangements of the four books that have book number 2 somewhere to the right of book number 1. □

c. *[The brute force approach.]* A sequence of tosses that ends with a second head and has two tails before that second head must have length four. There are $2^4 = 16$ such sequences, which are all equally likely if the coin is fair, so each sequence has a probability of $\frac{1}{16} = 0.0625$ of coming up. Three sequences have a head last and two tails and a head before that second head: $HTTH$, $THTH$, and $TTHH$. It follows that

$$\begin{aligned} P(\text{exactly two } T\text{s before the second } H) &= P(HTTH) + P(THTH) + P(TTHH) \\ &= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16} = 0.1875\,. \quad \square \end{aligned}$$

c. *[The plug-into-a-formula approach.]* Since we are asking for the probability of a certain number of tosses (since we have a specified number failures, *i.e.* Ts) before the kth success, we are dealing with a negative binomial distribution with $x = 4$ trials and $k = 2$ successes, where $p = q = \frac{1}{2}$. Using the formula for the probability function of the negative binomial distribution, as found on p. 187 of the textbook:

$$\begin{aligned} P(\text{exactly two } T\text{s before the second } H) &= \binom{x-1}{k-1} p^k q^{x-k} = \binom{4-1}{2-1} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} \\ &= \binom{3}{1} \cdot \frac{1}{4} \cdot \frac{1}{4} = 3 \cdot \frac{1}{16} = \frac{3}{16} = 0.1875 \quad \square \end{aligned}$$

d. A sum of two integers is odd exactly when one of the two integers is odd and the other even; if both integers being added are odd, or both are even, the sum will be even. A standard die has six sides, half of which have odd numbers (1, 3, and 5) and half of which have even numbers (2, 4, and 6), so if you roll the die once, $P(\text{odd}) = P(\text{even}) = \frac{1}{2} = 0.5$. Since the two rolls are independent of one another, we have:

$$\begin{aligned} P(\text{odd sum}) &= P(\text{first roll odd and second even } \textit{or} \text{ first roll even and second odd}) \\ &= P(\text{first roll odd and second even}) + P(\text{first roll even and second odd}) \\ &= P(\text{odd}) \cdot P(\text{even}) + P(\text{even}) \cdot P(\text{odd}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad \blacksquare \end{aligned}$$

2. Do any *two* (2) of **a**–**c**. *[10 = 2 × 5 each]*

a. A standard 52-card deck is divided up randomly into four hands of thirteen cards each. What is the probability that each of the four hands has an ace?

b. Suppose the continuous random variable X has $f(x) = \begin{cases} \frac{1}{3} & \text{if } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$ as its probability density function. Compute $P(X \leq 2)$.

c. Suppose A and B are events in some sample space S. What is $P(A \cap B) + P\left(\bar{A} \cap B\right)$?

Solutions. **a.** First, note that there are $\binom{52}{13}$ ways to select the first hand of thirteen cards, $\binom{39}{13}$ ways to select the second hand, $\binom{26}{13}$ ways to select the third hand, and $\binom{13}{13} = 1$ ways to select the fourth hand. This gives a total of $\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$ ways to divide up the deck into four hands of thirteen cards each. Since the division of the deck into four hands of thirteen cards each is random, any such division is as likely as any other.

Second, we count the number of divisions in which each hand includes exactly one ace. There are $\binom{48}{12}$ ways to choose the rest of the hand that includes the A$\heartsuit$, then $\binom{36}{12}$ ways to choose the rest of the hand that includes the A$\diamondsuit$, then $\binom{24}{12}$ ways to choose the rest of the hand that includes the A$\clubsuit$, and then $\binom{12}{12} = 1$ ways to choose the rest of the hand that includes the the A$\spadesuit$. This gives a total of $\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}$ ways to divide up the deck into four hands of thirteen cards each so that each hand includes exactly one ace.

It follows that

$$\begin{aligned} P(\text{each hand has one ace}) &= \frac{\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}} = \frac{\frac{48!}{36!12!}\,\frac{36!}{24!12!}\,\frac{24!}{12!12!}\,\frac{12!}{0!12!}}{\frac{52!}{39!13!}\,\frac{39!}{26!13!}\,\frac{26!}{13!13!}\,\frac{13!}{0!13!}} \\ &= \frac{\frac{48!}{12!12!12!12!}}{\frac{52!}{13!13!13!13!}} = \frac{48!}{12!12!12!12!} \cdot \frac{13!13!13!13!}{52!} \\ &= \frac{13^4}{52 \cdot 51 \cdot 50 \cdot 49} \approx 0.0044\,. \quad \square \end{aligned}$$

b. By definition,

$$P(X \leq 2) = \int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} \frac{1}{3}\,dx = 0 + \frac{1}{3} \cdot x \Big|_0^2 = \frac{1}{3} \cdot 2 - \frac{1}{3} \cdot 0 = \frac{2}{3} \quad \square$$

c. Since $(A \cap B) \cap \left(\bar{A} \cap B\right) = \left(A \cap \bar{A}\right) \cap B = \emptyset \cap B = \emptyset$,

$$\begin{aligned} P(A \cap B) + P\left(\bar{A} \cap B\right) &= P\left((A \cap B) \cup \left(\bar{A} \cap B\right)\right) \\ &= P\left(\left(A \cup \bar{A}\right) \cap B\right) = P\left(S \cap B\right) = P(B)\,, \end{aligned}$$

where S is the entire sample space. ■

3. Do either *one* (1) of **a** or **b**. *[8]*

a. Suppose the continuous random variable Y has $g(y) = \begin{cases} \frac{3}{2}y^2 & \text{if } -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$ as its probability density function. Find the number m such that $P(Y \leq m) = \frac{9}{16}$.

b. A bin initially contains five blue and five red balls. Balls are drawn randomly, one at a time and without replacement, from the bin until it is empty. [$5 + 5 = 10$ draws.] Must any two sequences of 10 draws be equally likely? Explain why or why not.

SOLUTIONS. **a.** Since $g(y) > 0$ only for $-1 \leq y \leq 1$, m should be somewhere between -1 and 1. (An $m \leq -1$ would give $P(Y \leq m) = 0 < \frac{9}{16}$, and an $m \geq 1$ would give $P(Y \leq m) = 1 > \frac{9}{16}$.) For $-1 < m < 1$, we have:

$$\begin{aligned} P(Y \leq m) &= \int_{-\infty}^{m} g(y)\,dy = \int_{-\infty}^{-1} 0\,dy + \int_{-1}^{m} \frac{3}{2}y^2\,dy = 0 + \frac{3}{2} \cdot \frac{y^3}{3}\bigg|_{-1}^{m} \\ &= \frac{m^3}{2} - \frac{(-1)^3}{2} = \frac{m^3}{2} - \frac{-1}{2} = \frac{m^3}{2} + \frac{1}{2} \end{aligned}$$

We therefore need to solve for m in the equation $\dfrac{m^3}{2} + \dfrac{1}{2} = \dfrac{9}{16}$:

$$\begin{aligned} \frac{m^3}{2} + \frac{1}{2} = \frac{9}{16} &\implies \frac{m^3}{2} = \frac{9}{16} - \frac{1}{2} = \frac{9}{16} - \frac{8}{16} = \frac{1}{16} \\ &\implies m^3 = 2 \cdot \frac{1}{16} = \frac{1}{8} \implies m = \sqrt[3]{\frac{1}{8}} = \frac{1}{2} \quad \square \end{aligned}$$

b. Any two sequences of ten draws are equally likely, even though the probabilities of getting a ball of a given colour on a particular draw may vary widely between sequences, as well as between different draws in the same sequence. Consider, for example, the probabilities of two different sequences, laid out in terms of the probabilities of coosing each ball in the sequence:

$$\begin{aligned} P(RRBRBBBBRR) &= \frac{5}{10} \cdot \frac{4}{9} \cdot \frac{5}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{2}{2} \cdot \frac{1}{1} \\ P(BRRRBBRBRB) &= \frac{5}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} \end{aligned}$$

As you compute the probabilities of choosing the next ball in the sequences, the denominators count down from 10 to 1, reflecting the number of balls left to choose from, while the numerators do two mixed together countdowns from 5 to 1, reflecting the number of balls left of each colour. It is not hard to see that it follows that

$$P(\text{any sequence of 10 draws}) = \frac{5!5!}{10!} \approx 0.0040\,. \quad \blacksquare$$

[Total = 30]