Mathematics 1550H – Introduction to probability
Trent University, Winter 2015
Solutions to the Quizzes

Quiz #1. *Thursday, 15 January, 2015. [10 minutes]*

1. The following problem appeared online some years ago. [Its origins are unknown to your instructor.]

> *If you choose an answer to this question at random, what is the chance you will be correct?*
>
> *A) 25%*
> *B) 50%*
> *C) 60%*
> *D) 25%*

Explain, as completely (and correctly!) as you can just what is going on in this problem. *[5]*

Solution. Sadly, there is no correct answer among the given alternatives, so the probability of choosing a correct answer is 0%. A random choice of four different answers, exactly one of which is correct, has a 25% chance (*i.e.* a probability of $\frac{1}{4} = 0.25$) of being correct. In this case, however, two of the four alternatives are the same, namely 25%, so the possible answer of 25% would have a 50% chance (*i.e.* a probability of $\frac{1}{2} - 0.5$) of being chosen. To add insult to injury, the possible answer of 50% occurs once, so it would be randomly chosen 25% of the time (*i.e.* with a probability of $\frac{1}{4} = 0.25$).

Essentially, the problem has been set up to be internally contradictory. What makes this possible is the fact that the problem is self-referential: the opening phrase "If you choose an answer to this question at random" makes the question refer to itself, and then the evil selection of possible answers makes it impossible to find a correct answer. ■

Quiz #2. *Thursday, 22 January, 2015. [10 minutes]*

1. A fair standard six-sided die is rolled twice. What is the probability that at least one of the two rolls came up with an odd number?

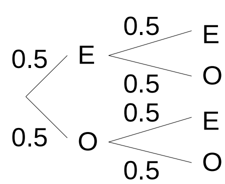
Solution. The probability that that at least one of the two rolls came up with an odd number is $\frac{3}{4} = 0.75$. A single roll of a fair standard six-sided die can come up in six ways, with 1, 2, 3, 4, 5, or 6 on the upward face. Half of these are even, namely 2, 4, and 6, and the other half are odd, namely 1, 3, and 5. Since the die is fair, each face is as likely to come up as any other, so each face has a probability of $\frac{1}{6}$ of coming up on a given roll. As the die has three odd and three even faces out of a total of six faces, for a single roll of the die we have:

$$P(\text{odd}) = P(\text{even}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2} = 0.5$$

With two rolls of the die, the only way to *not* get an odd number on at least one of the rolls is to roll an even on both rolls, which has a probability of $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 0.25$. It then follows that:

$$P(\text{at least one odd}) = 1 - P(\text{two even}) = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$$

There are, of course, other ways to tackle the problem. For example, one could trace the outcomes and probabilities in a suitable tree diagram such as:



■

Quiz #3. *Thursday, 29 January, 2015. [10 minutes]*

An edition of the *Rise and Fall of the Roman Vampire*† has seven volumes, numbered 1 to 7.

1. How many ways are there to arrange the seven volumes on three shelves? (Including the possibilities that one or two shelves have no volumes.) *[2.5]*
2. How many ways are there to choose a group of three volumes out of the seven if the group must contain exactly one even-numbered volume. (The order in which the volumes of the group are chosen does not matter.) *[2.5]*

SOLUTIONS. 1. We did a similar problem in class with fewer books and only two shelves. The same key idea works here, though: think of this in terms of arranging the seven volumes plus two dividers. The books before the first divider go on the first shelf, the books between the dividers go on the second shelf, and the books after the second divider go on the third shelf.

An arrangement of seven books and two dividers has nine positions. There are then

$$\binom{9}{2} = \frac{9!}{(9-2)!2!} = \frac{9!}{7!2!} = \frac{9 \cdot 8}{2 \cdot 1} = 9 \cdot 4 = 36$$

ways of picking the two positions for the dividers, and $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways of arranging the seven volumes in the remaining seven positions. It follows that there are

$$\binom{9}{2} \cdot 7! = 36 \cdot 5040 = 181440$$

ways to arrange the seven volumes on three shelves. [It's perfectly OK with me to leave the answer in the unsimplified form $\binom{9}{2} \cdot 7! \dots$] $\Box$

2. The list of numbers 1, 2, 3, 4, 5, 6, 7, includes three even and four odd numbers among the seven numbers in the list. To pick a group of three numbers from the list that includes exactly one even number, we pick one of the three even numbers – there are $\binom{3}{1} = 3$ ways to do so – and pick two numbers out of the four odd numbers on the list – there are $\binom{4}{2} = 6$ ways to do so. It follows that there are

$$\binom{3}{1}\binom{4}{2} = 3 \cdot 6 = 18$$

ways to choose a group of three volumes out of the seven if the group must contain exactly one even-numbered volume. [Again, it's OK to leave the answer in the unsimplified form $\binom{3}{1}\binom{4}{2}$.] $\blacksquare$

† With apologies to the perpetrators of the Bugs Bunny cartoon *Transylvania 6-5000*.

Quiz #4. *Thursday, 5 February, 2015. [15 minutes]*

Do *one* (1) of the following questions.

1. Four cards are drawn, one at a time and without replacement from a standard 52-card deck. Let A be the event that the four cards are of different suites, so each suite occurs once among the four cards, and B is the event that all four cards are of the same kind. What are $P(A|B)$ and $P(B|A)$?
2. Suppose S is a sample space and A and B are events such that $A \cup B = S$ and $P(A) = P(B) = \frac{5}{8} = 0.625$. What is $P(A|B)$? *[5]*

SOLUTIONS. 1. Note that if all four cards are of the same kind, then they must be of different suites, so the event B is a subset of the event A, *i.e.* $B \subseteq A$. It follows that $A \cap B = B$, so $P(A \cap B) = P(B)$, and thus $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$.

To compute $P(B|A)$, we'll have to work a little harder, though we can still exploit the observation that $A \cap B = B$ and $P(A \cap B) = P(B)$. We will need to know $P(A)$ and $P(B)$. There are $52 \cdot 51 \cdot 50 \cdot 49 = \frac{52!}{(52-48)!}$ ways to choose four cards, one at a time and without replacement, from a standard 52-card deck. [This is the size of our sample space, and every outcome is equally likely.]

Event B, that all four cards are of the same kind, has $52 \cdot 3 \cdot 2 \cdot 1 = 52 \cdot 3!$ outcomes in it: there are 52 ways to choose the first card, and, since there are four cards of each kind, 3 ways to choose a second card of the same kind, 2 ways to choose a third card of the same kind, and 1 way to choose a fourth card of the same kind. It follows that $P(B) = \frac{\#\text{ of outcomes in } B}{\#\text{ of outcomes}} = \frac{52 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49}$. [We will simplify later, when we're computing $P(B|A)$.]

Event A, that the four cards are of different suites, has $52 \cdot 39 \cdot 26 \cdot 13$ outcomes in it: there are are 52 ways to choose the first card, $36 = 3 \cdot 13$ ways to choose a second card of a different suite from the first card, $26 = 2 \cdot 13$ ways to choose a third card of a different suite from the first two cards, and $13 = 1 \cdot 13$ ways to choose a fourth card from a suite different from the other three cards. It follows that $P(A) = \frac{\#\text{ of outcomes in } A}{\#\text{ of outcomes}} = \frac{52 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49}$. [Again, we will simplify later, when we're computing $P(B|A)$.]

It remains to compute $P(B|A)$. Since, as noted above, $P(A \cap B) = P(B)$,

$$\begin{aligned}
P(B|A) &= \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{\frac{52 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49}}{\frac{52 \cdot 39 \cdot 26 \cdot 13}{52 \cdot 51 \cdot 50 \cdot 49}} = \frac{52 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49} \cdot \frac{52 \cdot 51 \cdot 50 \cdot 49}{52 \cdot 39 \cdot 26 \cdot 13} \\
&= \frac{52 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 39 \cdot 26 \cdot 13} = \frac{3 \cdot 2 \cdot 1}{39 \cdot 26 \cdot 13} = \frac{3 \cdot 2 \cdot 1}{(3 \cdot 13) \cdot (2 \cdot 13) \cdot (1 \cdot 13)} = \frac{1}{13 \cdot 13 \cdot 13} \\
&= \frac{1}{13^3} = \frac{1}{2197} \approx 0.000455\,.
\end{aligned}$$

Note that $P(A|B)$ and $P(B|A)$ are wildly different ... ■

2. $P(A|B) = \frac{P(A \cap B)}{P(B)}$ by definition, and we are given that $P(B) = \frac{5}{8} = P(A)$, so we need to figure out $P(A \cap B)$. Recall that, in general, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and note that we have $P(A \cup B) = P(S) = 1$ because $A \cup B = S$. It follows that

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{5}{8} + \frac{5}{8} - 1 = \frac{5}{4} - 1 = \frac{1}{4} = 0.25\,,$$

and so

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{5}{8}} = \frac{1}{4} \cdot \frac{8}{5} = \frac{2}{5} = 0.4\,.$$

It's worth noticing that because of the symmetry in the set-up, $P(B|A) = P(A|B) = 0.4$. ■

Quiz #5. *Thursday, 12 February, 2015. [15 minutes]*
Suppose X is a continuous random variable with probability density function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \text{ or } x > 1 \\ 2x & \text{if } 0 \leq x \leq 1 \end{cases} .$$

Let A be the event that $X \leq \frac{1}{2}$ and B be the event that $X \geq \frac{1}{4}$.

1. Compute $P(A)$. *[2]*
2. Compute $P(B|A)$. *[3]*

SOLUTIONS. 1. Recalling that the antiderivative of x is $\frac{x^2}{2}$ [using the Power Rule for integration]. we have:

$$\begin{aligned} P(A) &= P(X \leq \frac{1}{2}) = \int_{-\infty}^{1/2} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1/2} 2x\,dx = 0 + 2 \cdot \left. \frac{x^2}{2} \right|_0^{1/2} \\ &= \left. x^2 \right|_0^{1/2} = \left(\frac{1}{2}\right)^2 - 0^2 = \frac{1}{4} - 0 = \frac{1}{4} = 0.25 \qquad \square \end{aligned}$$

2. By definition, $P(B|A) = \frac{P(B \cap A)}{P(A)}$. We computed $P(A)$ in anwering question 1 above, but we still need to compute $P(B \cap A)$. Note that since A is the event that $X \leq \frac{1}{2}$ and B is the event that $X \geq \frac{1}{4}$, and $A \cap B$ is the event that both A and B occur, $A \cap B$ is the event that $\frac{1}{4} \leq X \leq \frac{1}{2}$. It follows that

$$\begin{aligned} P(A \cap B) &= P\left(\frac{1}{4} \leq X \leq \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\,dx = \int_{1/4}^{1/2} 2x\,dx = 2 \cdot \left. \frac{x^2}{2} \right|_{x/4}^{x/2} \\ &= \left. x^2 \right|_{x/4}^{x/2} = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16} = 0.1875\,. \end{aligned}$$

It remains to actually compute $P(B|A)$:

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{3/16}{1/4} = \frac{3}{4} = 0.75 \qquad \blacksquare$$

Quiz #6. *Take-Home! [Due in class on Thursday, 12 March.]*

Meredith Mortiser wishes to use a saw to cut a wooden cube, with sides 30 *cm* long, into 27 cubes, each with sides 10 *cm* long. Meredith can do this easily by making six cuts through the cube, keeping the pieces together in the cube shape.

$-e^{i\pi}$. Can Meredith reduce the number of necessary cuts by rearranging the pieces after each cut? If so, how? If not, why not? *[5]*

SOLUTION. Meredith must use at least six cuts, even if (s)he tries to rearrange the pieces at intermediate stages. To see this observe that the center cube of the 27 small ones the larger cube is cut up into has – like every other cube – six faces. Each of those six faces requires a separate cut, because there is no way to rearrange the pieces at any stage to allow two or more faces to result fro a single cut. ■

Quiz #7. ~~*Thursday, 12*~~ *Tuesday, 17 March, 2015. [15 minutes]*

A fair coin is tossed three times, and X is the number of heads that occur.

1. What is the probability function $p(x) = P(X = x)$ of the random variable X? *[3]*
2. Compute the expected value $E(X)$ of X. *[2]*

SOLUTIONS. 1. Tossing a fair coin three times could result in 0, 1, 2, or 3 heads, in several ways for the middle two. Since the coin is fair, any sequence of three tosses is as likely as any other; since there are $2^3 = 8$ possible sequences, each has probability $\frac{1}{8} = 0.125$, and so:

$$\begin{aligned}
p(0) &= P(X = 0) = P(TTT) = \frac{1}{8} = 0.125 \\
p(1) &= P(X = 1) = P(HTT) + P(THT) + P(TTH) = \frac{3}{8} = 0.375 \\
p(2) &= P(X = 2) = P(HHT) + P(HTH) + P(THH) = \frac{3}{8} = 0.375 \\
p(3) &= P(X = 3) = P(HHH) = \frac{1}{8} = 0.125
\end{aligned}$$

That is, $p(x) = \begin{cases} \frac{1}{8} & \text{if } x = 0 \text{ or } x = 3 \\ \frac{3}{8} & \text{if } x = 1 \text{ or } x = 2 \\ 0 & \text{otherwise} \end{cases}$. □

2. By definition, $E(X) = \sum xp(x)$, where x ranges over all possible values of X. Applying the definition in this case gives us:

$$E(X) = \sum_{x=0}^{3} xp(x) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2} = 1.5 \quad ■$$

Quiz #8. *Thursday, 19 March, 2015. [15 minutes]*

Let X be a continuous random variable with density function $f(x) = \begin{cases} \frac{3}{4}\left(1-x^2\right) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$.

1. Compute the expected value $E(X)$ of X. *[3]*
2. Compute the variance $V(X)$ of X. *[2]*

Solutions. 1. By definition, given that X is a continuous random variable:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{-1} x \cdot 0\,dx + \int_{-1}^{1} x \cdot \frac{3}{4}\left(1-x^2\right)\,dx + \int_{1}^{\infty} x \cdot 0\,dx \\ &= 0 + \frac{3}{4}\int_{-1}^{1}\left(x - x^3\right)\,dx + 0 = \frac{3}{4}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{-1}^{1} = \frac{3}{4}\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \frac{3}{4}\left(\frac{(-1)^2}{2} - \frac{(-1)^4}{4}\right) \\ &= \frac{3}{4}\cdot\frac{1}{4} - \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{16} - \frac{3}{16} = 0 \qquad \square \end{aligned}$$

2. Again, by definition, and using the fact obtained above that $E(X) = 0$:

$$\begin{aligned} V(X) &= E\left([X - E(X)]^2\right) = E\left(X^2\right) - [E(X)]^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - 0^2 \\ &= \int_{-\infty}^{-1} x^2 \cdot 0\,dx + \int_{-1}^{1} x^2 \cdot \frac{3}{4}\left(1-x^2\right)\,dx + \int_{1}^{\infty} x^2 \cdot 0\,dx = 0 + \frac{3}{4}\int_{-1}^{1}\left(x^2 - x^4\right)\,dx + 0 \\ &= \frac{3}{4}\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{-1}^{1} = \frac{3}{4}\left(\frac{1^3}{3} - \frac{1^5}{5}\right) - \frac{3}{4}\left(\frac{(-1)^3}{3} - \frac{(-1)^5}{5}\right) = \frac{3}{4}\cdot\frac{2}{15} - \frac{3}{4}\cdot\frac{-2}{15} \\ &= \frac{1}{10} + \frac{1}{10} = \frac{2}{10} = \frac{1}{5} = 0.2 \qquad \blacksquare \end{aligned}$$

Quiz #9. *Thursday, 26 March, 2015. [10 minutes]*

1. A forest is populated by trees whose ages have a mean of $\mu = 30$ years and a standard deviation of $\sigma = 4$ years. What is the maximum possible value of the probability that a randomly chosen tree is either no more than 25 years or not less than 35 years old? *[5]*

Solution. Let the random variable X be the age of the randomly chosen tree. We are given that $\mu = E(X) = 30$ and $\sigma^2 = V(X) = 4^2 = 16$. Note that saying "$X \le 25$ or $X \ge 35$" is equivalent to saying that $|X - 30| \ge 5$. By Chebyshev's Inequality, *i.e.* $P\left(|X - \mu| \ge t\right) \le \dfrac{\sigma^2}{t^2}$, it then follows that:

$$P(X \le 25 \text{ or } X \ge 35) = P\left(|X - 30| \ge 5\right) = \frac{4^2}{5^2} = \frac{16}{25} = 0.64$$

Thus the probability that a randomly chosen tree is either no more than 25 years or not less than 35 years old is at most $\frac{16}{25} = 0.64$. $\blacksquare$