Mathematics 1550H – Introduction to probability
Trent University, Winter 2015

Final Examination
Friday, 10 April, 2015

Time: 3 hours *(14:00-17:00 at the AC)* *Brought to you by* Стефан Біланюк.

Instructions: Do both of parts **H** and **T**, and, if you wish, part **E**. Show all your work and simplify answers as much as practicable. *If in doubt about something,* **ask!**

Aids: Calculator; one $8.5'' \times 11''$ or A4 aid sheet; standard normal table; ≤ 1 brain.

Part H. Do all of **1**–**5**. *[Subtotal = 70/100]*

1. A fair coin is tossed four times. Let A be the event that the number of heads is odd, and let B be the event that more than two heads occur.

a. What are the sample space and probability function? *[5]*

b. Compute $P(A)$ and $P(B)$. *[5]*

c. Compute $P(A|B)$ and determine whether A and B are independent or not. *[5]*

2. Suppose X is a normally distributed continuous random variable with mean $\mu = -5$ and standard deviation $\sigma = 10$.

a. Compute $P(X \leq 0)$. *[5]*

b. Find the value of c such that $P(X > c) \approx 0.2266$. *[5]*

3. A hand of five cards is drawn simultaneously (without order or replacement) from a standard 52-card deck.

a. What is the probability that the hand is a *busted flush,* that is, that four of the cards are from the same suit and the remaining card is from a different suit? *[8]*

b. What is the probability that there is at least one card from each of the four suits in the hand? *[7]*

4. A fair coin is tossed until it comes up again with one or the other face for the second time.

a. What are the sample space and probability function? *[7]*

b. Let A be the event that three tosses took place and let B be the event that the first toss was a head. Determine whether the events A and B are independent or not. *[8]*

5. Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{3}{x^4} & x \geq 1 \\ 0 & x < 1 \end{cases} = \begin{cases} 3x^{-4} & x \geq 1 \\ 0 & x < 1 \end{cases} .$$

Compute the expected value $E(X)$ and variance $V(X)$ of X. *[15]*

*[Parts **T** and **E** are on page 2.]*

Part T. Do any *two* (2) of **6–10**. *[Subtotal = 30/100]*

6. A box initially contains six balls, three red and three green. Balls are drawn randomly from the box, without replacement, until the second time a red ball is drawn. Let X be the number of draws made during the process.

a. Find the probability function of X. *[6]*

b. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[4]*

c. Let A be the event that $X = 3$ and let B be the event that the first ball drawn is red. Determine whether A and B are independent events or not. *[5]*

7. Suppose the discrete random variables X and Y are jointly distributed according to the following table:

$X \backslash Y$	1	2	3
1	0.2	0.2	0
2	0.1	0	0.2
3	0	0.1	0.2

a. Compute the expected values $E(X)$ and $E(Y)$, variances $V(X)$ and $V(Y)$, and covariance $\mathrm{Cov}(X, Y)$ of X and Y. *[10]*

b. Let $U = 3X + Y$. Compute $E(U)$ and $V(U)$. *[5]*

8. Let $g(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ be the probability density function of the continuous random variable X.

a. Verify that $g(x)$ is indeed a probability density function. *[7]*

b. Compute the expected value $E(X)$ and variance $V(X)$ of X. *[8]*

9. Suppose Y is a continuous random variable with an exponential distribution such that $P(Y \leq \ln(2)) = \frac{1}{2} = 0.5$.

a. Determine the expected value, $E(Y)$, of Y. *[10]*

b. Use Markov's Inequality to estimate $P(Y \geq 4)$ and then compute it exactly. *[5]*

10. Suppose W is a discrete random variable with possible values 0, 1, 2, ... , and such that $E(W) = V(W) = 1$. Show that for any integer $k \geq 1$, $P(W \geq k + 1) \leq \dfrac{1}{k^2}$. *[15]*

[Total = 100]

Part E. Bonus!

∘ . Give an example of jointly distributed random variables X and Y which are not independent, but for which $E(XY) = E(X)E(Y)$ nevertheless. *[2]*

∘∘∘ ∘∘∘ . Write an original little poem about probability or mathematics in general. *[2]*

I HOPE THAT YOU ENJOYED THE COURSE! HAVE A GOOD SUMMER!