TRENT UNIVERSITY, SUMMER 2023 (S62)

MATH 1550H Test Solutions

Monday, 10 July

Time: 50 minutes

Name: Ya Nikhto

STUDENT NUMBER: -0000000

Question	**Mark**	
1	___	
2	___	
3	___	
Total	___	/30

Instructions

- *Show all your work.* Legibly, please!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- If you do more parts of a question than were asked for, cross out the one you do not want marked; otherwise the marker will not consider the part they encounter last.
- You may use a calculator, so long as it cannot communicate with other devices, and one letter- or A4-size aid sheet with whatever you want written on all sides of it.

1. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. How many different ways are there to arrange all the letters in the word "unusual" if the three copies of "u" cannot be told apart?

b. A biased coin with $P(H) = 0.6$ and $P(T) = 0.4$ is tossed three times. What are the sample space and probability distribution function for this experiment?

c. A five-card hand is drawn from a standard 52-card deck, without order or replacement. What is the probability that the cards in the hand are all of different kinds (*i.e.* ranks)?

SOLUTIONS. **a.** "Unusual" has seven letters, so there would be 7! ways to arrange them if they were all distinguishable. However, we cannot tell the three copies of "u" apart, so we have to divide by the number of ways three things can be arranged. Thus, there are $\frac{7!}{3!} = \frac{5040}{6} = 840$ different ways to arrange all the letters in the word "unusual" if the three copies of "u" cannot be told apart. □

b. The sample space consists of all sequences length three of heads and/or tails:

$$S = \{\, HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \,\}$$

Since $P(H) = 0.6$ and $P(T) = 0.4$, the probability distribution function, which we'll call m for once as a nod to the textbook, is given by:

$$\begin{aligned}
m(HHH) &= 0.6^3 = 0.216 \\
m(HHT) = m(HTH) = m(THH) &= 0.6^2 \cdot 0.4 = 0.144 \\
m(HTT) = m(THT) = m(TTH) &= 0.6 \cdot 0.4^2 = 0.096 \\
m(TTT) &= 0.4^3 = 0.064 \qquad \square
\end{aligned}$$

c. There are thirteen kinds or ranks in the deck, and four of each kind (one for each suit), so there are $\binom{13}{5}$ ways to choose five different kinds to be represented in a hand, and $\binom{4}{1}$ to choose one of the four cards of that kind for each of the five kinds. It follows that there are

$$\binom{13}{5}\binom{4}{1}^5 = 1287 \cdot 1024 = 1317888$$

five-card hands in which card is of a different kind. Since each hand is as likely to be drawn as any other, and there are $\binom{52}{5} = 2598960$ possible five-card hands in total, it follows that the probability that the cards in a randomly drawn five-card hand all being of different kinds is:

$$\frac{\binom{13}{5}\binom{4}{1}^5}{\binom{52}{5}} = \frac{1317888}{2598960} \approx 0.5071 \qquad \square$$

2. Do any *two (2)* of **a**–**c**. *[10 = 2 × 5 each]*

a. The continuous random variable X has $f(x) = \begin{cases} x/8 & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$ as its probability density function. Compute $P(X \geq 1 \mid X \leq 3)$.

b. A biased coin with $P(H) = 0.6$ and $P(T) = 0.4$ is tossed three times. The random variable Y counts the number of heads that come up in those three tosses. Compute the expected value of Y.

c. Suppose $g(x) = \begin{cases} c\left(1-x^2\right) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$, where c is a constant. Find the value of the constant c which would make $g(x)$ a valid probability density function.

SOLUTIONS. **a.** By definition, $P(X \geq 1 \mid X \leq 3) = \dfrac{P(X \geq 1 \;\&\; X \leq 3)}{P(X \leq 3)} = \dfrac{P(1 \leq X \leq 3)}{P(X \leq 3)}$, so we need to compute $P(1 \leq X \leq 3)$ and $P(X \leq 3)$.

$$P(1 \leq X \leq 3) = \int_1^3 f(x)\,dx = \int_1^3 \frac{x}{8}\,dx = \frac{1}{8} \cdot \frac{x^2}{2}\bigg|_1^3 = \frac{3^2}{16} - \frac{1^2}{16} = \frac{9-1}{16} = \frac{8}{16} = \frac{1}{2}$$

$$\begin{aligned} P(X \leq 3) &= \int_{-\infty}^3 f(x)\,dx = \int_{-\infty}^0 0\,dx + \int_0^3 \frac{x}{8}\,dx = 0 + \frac{1}{8} \cdot \frac{x^2}{2}\bigg|_0^3 = \frac{x^2}{16}\bigg|_0^3 \\ &= \frac{3^2}{16} - \frac{0^2}{16} = \frac{9}{16} - 0 = \frac{9}{16} \end{aligned}$$

Thus $P(X \geq 1 \mid X \leq 3) = \dfrac{P(1 \leq X \leq 3)}{P(X \leq 3)} = \dfrac{\frac{1}{2}}{\frac{9}{16}} = \dfrac{1}{2} \cdot \dfrac{16}{9} = \dfrac{8}{9} \approx 0.8889$. $\square$

b. Note that the underlying setup is the same as in **1b**. The possible values of Y are 0, 1, 2, and 3. Their probabilities are:

$$\begin{aligned} P(Y=0) &= P(TTT) = 0.4^3 = 0.064 \\ P(Y=1) &= P\left(\{\,HTT,\ THT,\ TTH\,\}\right) = 3 \cdot 0.4^2 \cdot 0.6 = 3 \cdot 0.096 = 0.288 \\ P(Y=2) &= P\left(\{\,HHT,\ HTH,\ THH\,\}\right) = 3 \cdot 0.4 \cdot 0.6^2 = 3 \cdot 0.144 = 0.432 \\ P(Y=3) &= P(HHH) = 0.6^3 = 0.216 \end{aligned}$$

It follows, by the definition of expected value for discrete random variables, that the expected value of Y is:

$$\begin{aligned} E(Y) &= \sum_{k=0}^{3} kP(Y=k) = 0 \cdot 0.064 + 1 \cdot 0.288 + 2 \cdot 0.432 + 3 \cdot 0.216 \\ &= 0 + 0.288 + 0.864 + 0.648 = 1.8 \qquad \square \end{aligned}$$

c. To be a valid density function, $g(x)$ must satisfy two conditions, from each of which we can get some information about what c would have to be.

First, we need to have $g(x) \geq 0$ for all x. For $x < -1$ or $x > 1$ we have $g(x) = 0 \geq 0$ by the definition of $g(x)$. We also need to gave $g(x) = c\left(1 - x^2\right) \geq 0$ when $-1 \leq x \leq 1$. Since $1 - x^2 \geq 0$ when $-1 \leq x \leq 1$, this means that we must have $c \geq 0$.

Second, we need to have that $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx = 1$. Let's work the integral out and see what value(s) of c would make it equal 1.

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\,dx &= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} c\left(1 - x^2\right)\,dx + \int_{1}^{\infty} 0\,dx = 0 + c\int_{-1}^{1} \left(1 - x^2\right)\,dx + 0 \\
&= c\left(x - \frac{x^3}{3}\right)\Bigg|_{-1}^{1} = c\left(1 - \frac{1^3}{3}\right) - c\left((-1) - \frac{(-1)^3}{3}\right) \\
&= c\left(1 - \frac{1}{3}\right) - c\left(-1 - \frac{-1}{3}\right) = c \cdot \frac{2}{3} - c \cdot \left(-\frac{2}{3}\right) = \frac{2}{3}c + \frac{2}{3}c = \frac{4}{3}c
\end{aligned}$$

Thus $\displaystyle\int_{-\infty}^{\infty} g(x)\,dx = \frac{4}{3}c = 1$ requires that $c = 1/\frac{4}{3} = \frac{3}{4} = 0.75$. As $c = \frac{3}{4} > 0$, this also makes the function satisfy $g(x) \geq 0$ for all x.

It follows that $g(x) = \begin{cases} c\left(1 - x^2\right) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ is a valid probability density function rxactly when $c = \frac{3}{4} = 0.75$. $\square$

3. Do *one (1)* of **a** or **b**. *[10]*

a. Suppose A and B are independent events in some sample space S, with both having positive probability. Does it have to be the case that $\bar{A} = \{\, s \in S \mid s \notin A \,\}$ is independent of B? If so, explain why; if not, give an example in which A and B are independent, but $\bar{A}$ and B are not.

b. A doctor gives a patient a test for a condition which occurs in the 1% of the population. Experience has shown that the test returns positive in 99% of the time and negative 1% of the time if the condition is present, and returns positive 5% of the time and negative 95% of the time if the condition is not present. If the test given by the doctor comes back positive, how likely is it that the patient actually has condition?

SOLUTIONS. **a.** Suppose A and B are independent, *i.e.* $P(A \cap B) = P(A)P(B)$,

Recall that $P\left(\bar{A}\right) = 1 - P(A)$. Also, since $A \cap B$ and $\bar{A} \cap B$ have nothing in common (as A and $\bar{A}$ have nothing in common), but make up B between them (as every outcome in B is either in A or not), it follows that $P(B) = P\left(\bar{A} \cap B\right) + P(A \cap B)$, so $P\left(\bar{A} \cap B\right) = P(B) - P(A \cap B)$.

It follows from these facts that

$$\begin{aligned} P\left(\bar{A} \cap B\right) &= P(B) - P(A \cap B) = P(B) - P(A)P(B) = (1 - P(A))\, P(B) \\ &= P\left(\bar{A}\right) P(B)\,, \end{aligned}$$

which means that $\bar{A}$ and B are independent. □

b. This is a job for Bayes' Theorem.

Let C be the event that the patient has the condition, and then $\bar{C}$ is the event that the patient does not have the condition. Since the condition occurs in 1% of the population, we have $P(C) = 0.01$ and $P\left(\bar{C}\right) = 1 - 0.01 = 0.99$.

Let $+$ be the event that that the test gives a positive result and let $-$ be the event that it gives a negative results. We are given that $P\left(+|C\right) = 0.99$, $P\left(-|C\right) = 0.01$, $P\left(+|\bar{C}\right) = 0.05$, and $P\left(-|\bar{C}\right) = 0.95$.

We want to know $P\left(C|+\right)$. Bayes' Theorem tells us that:

$$\begin{aligned} P\left(C|+\right) &= \frac{P\left(+|C\right) P(C)}{P\left(+|C\right) P(C) + P\left(+|\bar{C}\right) P\left(\bar{C}\right)} = \frac{0.99 \cdot 0.01}{0.99 \cdot 0.01 + 0.05 \cdot 0.99} \\ &= \frac{0.99 \cdot 0.01}{0.99(0.01 + 0.05)} = \frac{0.01}{0.01 + 0.05} = \frac{0.01}{0.06} = \frac{1}{6} \approx 0.1667 \qquad \square \end{aligned}$$

[Total = 30]