Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2023 (S62)

Solutions to Quiz #9
Reverse Engineering?

Instructions: Do all of the following problems. Please show all your work.

1. Suppose X is a continuous random variable. Find an example of a probability density function for X giving expected value $E(X) = 1$ and variance $V(X) = 3$ if X has ...

a. a uniform distribution. *[1]*
b. an exponential distribution. *[1]*
c. a normal distribution. *[1]*

In each case, if there is no such probability density function, explain why this is so.

SOLUTIONS. **a.** Here is an example: $f(x) = \begin{cases} \frac{1}{6} & -2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$. This is the density of a continuous uniform distribution with $a = -2$ and $b = 4$ (note that $b - a = 4 - (-2) = 6$). Consulting our handy-dandy little reference [1], it has expected value $E(X) = \dfrac{a+b}{2} = \dfrac{-2+4}{2} = \dfrac{2}{2} = 1$ and variance $V(X) = \dfrac{(b-a)^2}{12} = \dfrac{(4-(-2))^2}{12} = \dfrac{6^2}{12} = \dfrac{36}{12} = 3$, as desired. □

b. There is no exponential distribution meeting these conditions. If X has an exponential distribution with parameter λ, then $E(X) = \dfrac{1}{\lambda}$ and $V(X) = \dfrac{1}{\lambda^2} = [E(X)]^2$. Since $3 \neq 1 = 1^2$, it is impossible to have an exponential distribution with expected value $E(X) = 1$ and variance $V(X) = 3$. □

c. This one is a giveaway: you can have a normal distribution with any specified expected value and variance, so long as the latter is positive. In this case, we want to have $\mu = E(X) = 1$ and $\sigma = \sqrt{V(X)} = \sqrt{3}$. Consulting our handy-dandy little reference [1], the normal distribution with these parameters has density function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{\sqrt{3}\cdot\sqrt{2\pi}} e^{-(x-1)^2/2\left(\sqrt{3}\right)^2} = \frac{1}{\sqrt{6\pi}} e^{-(x-1)^2/6} \qquad \blacksquare$$

2. Is it possible to have a uniform probability distribution and an exponential probability distribution such that the continuous random variables with these distributions would have the same expected value and variance? Give an example demonstrating that this is possible or an explanation of why this is impossible. *[2]*

SOLUTION. It is indeed possible. Here is one example:

Let T be a continuous random variable with an exponential distribution with parameter $\lambda = 1$. Then, consulting our handy-dandy little reference [1], the expected value of T is $E(T) = \dfrac{1}{\lambda} = \dfrac{1}{1} = 1$ and the variance of T is $V(T) = \dfrac{1}{\lambda^2} = \dfrac{1}{1^2} = 1$.

Let X be a continuous random variable with a uniform distribution with left endpoint $a = 1 - \sqrt{3}$ and right endpoint $b = 1 + \sqrt{3}$. Then, consulting our handy-dandy little reference [1], the expected value of X is

$$E(X) = \frac{a+b}{2} = \frac{1 - \sqrt{3} + 1 + \sqrt{3}}{2} = \frac{2}{2} = 1$$

and the variance of X is

$$V(X) = \frac{(b-a)^2}{12} = \frac{\left((1+\sqrt{3}) - (1-\sqrt{3})\right)^2}{12} = \frac{\left(2\sqrt{3}\right)^2}{12} = \frac{2^2 \cdot 3}{12} = \frac{12}{12} = 1\,.$$

Thus the continuous random variables T and X, which have an exponential and a uniform distribution, respectively, have the same expected value and variance. □

Reference

1. *Some Common Probability Distributions – The Short Form*, MATH 1550H handout in the lecture of 2023-07-12.