Mathematics 1550H – Probability I: Introduction to Probability
Trent University, Summer 2023 (S62)

Solutions to Quiz #8
Recognition

Instructions: Do all of the following problems. Please show all your work.

All of the questions relate, in one way or another, to the following process, which uses a standard 52-card deck:

Step 1. Shuffle the deck throughly.
Step 2. Draw a card from the deck.
Step 3. Record which card was drawn.
Step 4. Replace the card in the deck.
Step 5. Go to step 1.

Yes, the process never ends ... :-) Some of the questions below, though, will only use some finite part of the process. Note that since the card drawn in each iteration of the process is replaced and then the deck is shuffled again before another card is drawn in the next iteration, the probabilities are the same for every draw.

1. The random variable H counts the number of $\heartsuit$s that turn up in the first 12 iterations of the process. What are $P(H=4)$, $E(H)$, and $V(H)$? *[1]*

Solution. In this case, each draw is effectively a Bernoulli trial where success is a drawing a $\heartsuit$. Thus the probability of success on each trial is $p = P(\heartsuit) = \frac{13}{52} = \frac{1}{4} = 0.25$ and probability of failure is $q = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$. H counts the successes in 12 of these trials, so H has a binomial distribution with $n = 12$ trials and p and q as above.

Consulting the handout summarizing some facts about various common distributions (see reference [1] below) tells us that

$$P(H=4) = \binom{12}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^{12-4} = \binom{12}{4}\frac{3^8}{4^{12}} = \frac{495 \cdot 6561}{16777216} = \frac{3247695}{16777216} \approx 0.1936\,,$$
$$E(H) = 12 \cdot \frac{1}{4} = 3\,, \quad \text{and} \quad V(H) = 12 \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{9}{4} = 2.25\,. \quad \square$$

2. The random variable Z returns a number for the card drawn on the forty-first iteration of the process: 0 if it is a $\heartsuit$, 1 if it is a $\diamondsuit$, 3 if it is a $\clubsuit$, and 4 if it is a $\spadesuit$. What are $P(1 \le Z \le 3)$, $E(Z)$, and $V(Z)$? *[1]*

Solution. In this case, we have a single draw with an equal likelihood of drawing a card from each of the four suits, since each suit has 13 cards in the deck. It follows that Z has an uniform distribution, with $P(Z=0) = P(Z=1) = P(Z=3) = P(Z=4) = \frac{1}{4} = 0.25$, although the values Z can take (0, 1, 3, or 4) are not equally spaced.

It now follows, from the respective definitions, that

$$
\begin{aligned}
P(1 \le Z \le 3) = P(Z=1) + P(Z=3) &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} = 0.5\,, \\
E(Z) &= 0P(Z=0) + 1P(Z=1) + 3P(Z=3) + 4P(Z=4) \\
&= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{0+1+3+4}{4} = \frac{8}{4} = 2\,, \\
E\left(Z^2\right) &= 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{4} + 3^2 \cdot \frac{1}{4} + 4^2 \cdot \frac{1}{4} = \frac{0+1+9+16}{4} = \frac{26}{4} = \frac{13}{2} = 6.5\,, \\
\text{and } V(Z) = E\left(Z^2\right) - [E(Z)]^2 &= \frac{13}{2} - 2^2 = \frac{13}{2} - 4 = \frac{13}{2} - \frac{8}{2} = \frac{5}{2} = 2.5\,. \quad \square
\end{aligned}
$$

3. The random variable W counts the number of iterations of the process required to have an ace (*i.e.* $A\heartsuit$, $A\diamondsuit$, $A\clubsuit$, or $A\spadesuit$) turn up for the first time. What are $P(W=4)$, $E(W)$, and $V(W)$? *[1]*

SOLUTION. In this case, we draw until an ace comes up and don't care what happens thereafter. Since there are 4 aces among the 52 cards in the deck, and each card is as likely to be drawn as any other, the probability of drawing an ace on any given draw is $P(A) = \frac{4}{52} = \frac{1}{13} \approx 0.769$. This means that W has a geometric distribution with probability of success $p = \frac{1}{13}$ and probability of failure $q = 1 - p = \frac{12}{13}$.

Consulting the handout summarizing some facts about various common distributions (see reference [1] below) tells us that

$$
P(W=4) = \left(\frac{12}{13}\right)^{4-1} \frac{1}{13} = \frac{12^3}{13^4} = \frac{1728}{28561} \approx 0.0605\,,\ E(W) = \frac{1}{\frac{1}{13}} = 13\,, \text{ and}
$$

$$
V(W) = \frac{\frac{12}{13}}{\left(\frac{1}{13}\right)^2} = \frac{12}{13} \cdot 13^2 = 12 \cdot 13 = 156\,. \quad \square
$$

4. The random variable D counts the number of iterations of the process required to have a $\diamondsuit$ turn up for the fourth time. What are $P(D=4)$, $E(D)$, and $V(D)$? *[1]*

SOLUTION. In this case, we keep iterating the process until the fourth time a $\diamondsuit$ comes up and don't care what happens after that. Each draw is effectively a Bernoulli trial where success is a drawing a $\diamondsuit$. There are 13 $\diamondsuit$s in the deck out of 52 cards, so the probability of success on each trial is $p = P(\diamondsuit) = \frac{13}{52} = \frac{1}{4} = 0.25$ and probability of failure is $q = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$. This means that D has a negative binomial distribution with the desired number of successes being $k = 4$ and the probabilities of success and failure as above.

Consulting the handout summarizing some facts about various common distributions (see reference [1] below) tells us that

$$
P(D=4) = \binom{4-1}{4-1} \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^{4-4} = \binom{3}{3} \cdot \frac{1}{4^4} \cdot \left(\frac{3}{4}\right)^0 = 1 \cdot \frac{1}{256} \cdot 1 = \frac{1}{256} \approx 0.0039\,,
$$

$$
E(D) = \frac{4}{\frac{1}{4}} = 4 \cdot 4 = 16\,, \text{ and } V(D) = \frac{4 \cdot \frac{3}{4}}{\left(\frac{1}{4}\right)^2} = \frac{3}{\frac{1}{16}} = 3 \cdot 16 = 48\,. \quad \square
$$

5. The random variable X_k, where $k \geq 1$, counts the number times a ♣ or ♠ turns up in in the $100(k-1)+1$st through the $100k$th iterations of the process. The random variable Y returns n if $X_n > 50$, but $X_k \leq 50$ for all k with $1 \leq k < n$. What are $P(Y=3)$, $E(Y)$, and $V(Y)$? *[1]*

SOLUTION. In this case, we have to work in stages: before we can deal with the random variable Y, we need to sort out how the random variables X_k work.

Since there are $13+13=26$ cards in the deck that are ♣ or ♠, the probability of getting one on each draw from the deck is $\frac{26}{52} = \frac{1}{2} = 0.5$, *i.e.* each draw is a Bernoulli trial with probability of success $p = \frac{1}{2}$, and hence also probability of failure $q = 1-p = 1 - \frac{1}{2} = \frac{1}{2} = 0.5$. Each X_k counts the number of successes in 100 of these Bernoulli trials, and so – consulting the handout again – has a binomial distribution with $P(X_k = n) = \binom{100}{n}\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{100-n} = \binom{100}{n}\frac{1}{2^{100}}$, where $n \in \{0, 1, \dots, n\}$.

We need to work out $P(X_k \leq 50)$ and $P(X_k > 50) = 1 - P(X_k \leq 50)$. This could be done directly, albeit with an unreasonable amount of work if done by hand. Still, with a little computer support, it's not too hard. Here is a way to compute $P(X_k \leq 50)$ using SageMath:

```
In [1]: var("k")
        sum( binomial(100,k), k, 0, 50 ) / 2^100

Out[1]: 171067743096724199353939462829/316912650057057350374175801344

In [2]: N(171067743096724199353939462829/316912650057057350374175801344)

Out[2]: 0.539794618693589
```

Rounding to a more manageable number of decimal places, this gives $P(X_k \leq 50) \approx 0.5398$ and $P(X_k > 50) = 1 - P(X_k \leq 50) \approx 1 - 0.5398 = 0.4602$.

An alternative approach to working out these probabilities, one more amenable to doing things by hand, is to exploit the general fact that $\binom{n}{i} = \binom{n}{n-i}$ as follows:

$$\begin{aligned}
P(X_k \leq 50) &= \sum_{i=0}^{50} P(X_k = i) = \sum_{i=0}^{50} \binom{100}{i} \cdot \frac{1}{2^{100}} = \left[\sum_{i=0}^{49} \binom{100}{i} \cdot \frac{1}{2^{100}}\right] + \binom{100}{50} \cdot \frac{1}{2^{100}} \\
&= \left[\sum_{i=0}^{49} \binom{100}{100-i} \cdot \frac{1}{2^{100}}\right] + \binom{100}{50} \cdot \frac{1}{2^{100}} \\
&= \left[\sum_{j=51}^{100} \binom{100}{j} \cdot \frac{1}{2^{100}}\right] + \binom{100}{50} \cdot \frac{1}{2^{100}} = P(X_k > 50) + \binom{100}{50} \cdot \frac{1}{2^{100}}
\end{aligned}$$

Since $P(X_k \leq 50) + P(X_k > 50) = 1$, it now follows that $2 \cdot P(X_k > 50) + \binom{100}{50} \cdot \frac{1}{2^{100}} = 1$, so $P(X_k > 50) = \frac{1 - \binom{100}{50} \cdot \frac{1}{2^{100}}}{2} = \frac{1}{2} - \binom{100}{50} \cdot \frac{1}{2^{101}} \approx 0.5 - 0.0398 = 0.4602$ [this still needs a really good calculator or a computer] and $P(X_k \leq 50) = 1 - P(X_k > 50) \approx 0.5398$.

Either way, or some other way, we have $P(X_k > 50) \approx 0.4602$ and $P(X_k \leq 50) \approx 0.5398$, which are respectively the probabilities of success or failure for the Bernoulli trial underlying Y. Since $Y = n$ for the first n for which $X_n > 50$, Y has a geometric distribution with $p \approx 0.4602$ and $q = 1 - p \approx 0.5398$. Consulting the handout summarizing some facts about various common distributions (see reference [1] below) tells us that

$$P(Y = 3) \approx 0.5398^{3-1} 0.4602 \approx 0.1341\,, \quad E(Y) \approx \frac{1}{0.4602} \approx 2.1730\,, \text{ and}$$
$$V(Y) \approx \frac{0.5398}{0.4602^2} \approx 2.5488\,. \quad \square$$

Reference

1. *Some Common Probability Distributions – The Short Form*, MATH 1550H handout in the lecture of 2023-07-12.