Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2023 (S62)

Solutions to Quiz #7
Variance

Instructions: Do the following problem. Please show all your work.

1. Two fair standard dice are rolled together and the random variable X returns the sum of the two faces that come up on the roll. Find the expected value and variance of X. *[5]*

SOLUTION. We'll work this out from scratch. With two fair standard dice being rolled, the sample space is $S = \{\, (a, b) \mid a,\ b \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}\,\}$, so there are 36 equally possible outcomes. The corresponding values of $X = a + b$ are given by the following table:

a\b	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

It follows that the possible outcomes of X are the integers 2 through 12, with probabilities given by

$$P(X = x) = \frac{\#\text{ outcomes adding up to } x}{36}\,,$$

giving us the following table:

x	2	2	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

By the definition of expected value for discrete random variables, the expected value of X is:

$$\begin{aligned}
E(X) &= \sum_{k=2}^{12} xP(X = x) \\
&= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} \\
&\qquad + 8 \cdot \frac{5}{36} + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} \\
&= \frac{2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12}{36} \\
&= \frac{252}{36} = 7
\end{aligned}$$

Similarly, the expected value of X^2 is:

$$\begin{aligned}
E\left(X^2\right) &= \sum_{k=2}^{12} x^2 P(X=x) \\
&= 2^2 \cdot \frac{1}{36} + 3^2 \cdot \frac{2}{36} + 4^2 \cdot \frac{3}{36} + 5^2 \cdot \frac{4}{36} + 6^2 \cdot \frac{5}{36} + 7^2 \cdot \frac{6}{36} \\
&\quad + 8^2 \cdot \frac{5}{36} + 9^2 \cdot \frac{4}{36} + 10^2 \cdot \frac{3}{36} + 11^2 \cdot \frac{2}{36} + 12^2 \cdot \frac{1}{36} \\
&= \frac{4 + 18 + 48 + 100 + 180 + 294 + 320 + 324 + 300 + 242 + 144}{36} \\
&= \frac{1974}{36} = \frac{329}{6} \approx 54.8333
\end{aligned}$$

It now follows, by definition, that the variance of X is:

$$V(X) = E\left(X^2\right) - \left[E(X)\right]^2 = \frac{329}{6} - 7^2 = \frac{329}{6} - 49 = \frac{329}{6} - \frac{294}{6} = \frac{35}{6} \approx 5.8333 \quad \square$$