Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2023 (S62)

Solutions to Quiz #6
Expected Values

Instructions: Do both of the following problems. Please show all your work.

1. A fair coin is tossed 5 times and the random variable X counts the number of heads that come in the 5 tosses. Use the definition of expected value for discrete random variables to compute the expected value of X. *[3]*

SOLUTION. Since the coin is fair the probability of is equal to the probability of a tail, *i.e.* $\frac{1}{2}$, on any given toss. In 5 tosses of this coin, the number of heads, *i.e.* the output of X, is one of the integers 0 through 5. If k is in this range, the probability of getting k heads, *i.e.* that $X = k$, is $P(X = k) = \binom{5}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{5-k} = \binom{5}{k}\left(\frac{1}{2}\right)^5$.

By the definition of expected value for discrete random variables, it follows that the expected value of X is:

$$\begin{aligned}
E(X) &= \sum_{k=0}^{5} kP(X=k) = \sum_{k=0}^{5} k\binom{5}{k}\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right)^5 \sum_{k=0}^{5} k\binom{5}{k} \\
&= \frac{1}{2^5}\left[0\binom{5}{0} + 1\binom{5}{1} + 2\binom{5}{2} + 3\binom{5}{3} + 4\binom{5}{4} + 5\binom{5}{5}\right] \\
&= \frac{1}{32}\left[0\cdot 1 + 1\cdot 5 + 2\cdot 10 + 3\cdot 10 + 4\cdot 5 + 5\cdot 1\right] \\
&= \frac{0+5+20+30+20+5}{32} = \frac{80}{32} = \frac{5}{2} = 2.5 \qquad \square
\end{aligned}$$

2. The continuous random variable T has $f(t) = \begin{cases} 2/t^3 & t \geq 1 \\ 0 & t < 1 \end{cases}$ as its probability density function. Use the definition of expected value for continuous random variables to compute the expected value of T. *[2]*

SOLUTION. By the definition of expected value for continuous random variables, the expected value of T is:

$$\begin{aligned}
E(T) &= \int_{-\infty}^{\infty} tf(t)\,dt = \int_{-\infty}^{1} t\cdot 0\,dt + \int_{1}^{\infty} t\cdot\frac{2}{t^3}\,dt = \int_{-\infty}^{1} 0\,dt + \int_{1}^{\infty}\frac{2}{t^2}\,dt \\
&= 0 + \int_{1}^{\infty} 2t^{-2}\,dt = 2\cdot\left.\frac{t^{-2+1}}{-2+1}\right|_1^{\infty} = \left.\frac{-2}{t}\right|_1^{\infty} = \frac{-2}{\infty} - \frac{-2}{1} = 0-(-2) = 2 \qquad \square
\end{aligned}$$