Mathematics 1550H – Probability I: Introduction to Probability
Trent University, Summer 2023 (S62)

Solutions to Quiz #4
Continuous Probability

Instructions: Do all of the following problems. Please show all your work.

Let $f(x) = \begin{cases} 1/x^2 & x \geq 1 \\ 0 & x < 1 \end{cases}$.

1. Verify that $f(x)$ is a valid probability density function. *[2]*

Solution. First, when $x < 1$, we have $f(x) = 0 \geq 0$, and when $x \geq 1$, $x^2 \geq 1 > 0$, so we have $f(x) = \frac{1}{x^2} > 0$. Thus $f(x) \geq 0$ for all x, satisfying the first condition for being a valid probability density function.

Second,

$$\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{1} 0\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx = 0 + \int_{1}^{\infty} x^{-2}\,dx = \left.\frac{x^{-2+1}}{-2+1}\right|_1^\infty$$
$$= -x^{-1}\Big|_1^\infty = \left.\frac{-1}{x}\right|_1^\infty = \frac{-1}{\infty} - \frac{-1}{1} = 0 - (-1) = 1\,,$$

so $f(x)$ also satisfies the second condition for being a valid probability density function.

Since it satisfies both of the necessary conditions, $f(x)$ is indeed a valid probability density function. □

Suppose that $f(x)$ (as given above) is the probability density function of some random process (technically, of a continuous random variable). Suppose $A = [-3, 3] \cup [6, 12]$ is the event that the process (or random variable) returns a value between -3 and 3 or between 6 and 12, and that $B = [2, 9]$ is the event that the process (or random variable) returns a value between 2 and 9.

2. Compute $P(B|A)$ and $P(A|B)$. *[3]*

Solution. Observe that $A \cap B = ([-3, 3] \cup [6, 12]) \cap [2, 9] = [2, 3] \cup [6, 9]$. To save a bit of effort, we will use the fact that $\frac{-1}{x}$ is the antiderivative of $\frac{1}{x^2}$ without further ado, having already worked it in solving question **1**. Here we go:

$$P(A) = \int_{-3}^{3} f(x)\,dx + \int_{6}^{12} f(x)\,dx = \int_{-3}^{1} 0\,dx + \int_{1}^{3} \frac{1}{x^2}\,dx + \int_{6}^{12} \frac{1}{x^2}\,dx$$
$$= 0 + \left.\frac{-1}{x}\right|_1^3 + \left.\frac{-1}{x}\right|_6^{12} = \frac{-1}{3} - \frac{-1}{1} + \frac{-1}{12} - \frac{-1}{6} = -\frac{1}{3} + 1 - \frac{1}{12} + \frac{1}{6}$$
$$= -\frac{4}{12} + \frac{12}{12} - \frac{1}{12} + \frac{2}{12} = \frac{-4 + 12 - 1 + 2}{12} = \frac{9}{12} = \frac{3}{4} = 0.75$$
$$P(B) = \int_{2}^{9} f(x)\,dx = \int_{2}^{9} \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_2^9 = \frac{-1}{9} - \frac{-1}{2} = -\frac{1}{9} + \frac{1}{2}$$
$$= -\frac{2}{18} + \frac{9}{18} = \frac{7}{18} \approx 0.3889$$

$$
\begin{aligned}
P(A \cap B) &= \int_2^3 f(x)\,dx + \int_6^9 f(x)\,dx = \int_2^3 \frac{1}{x^2}\,dx + \int_6^9 \frac{1}{x^2}\,dx = \left.\frac{-1}{x}\right|_2^3 + \left.\frac{-1}{x}\right|_6^9 \\
&= \frac{-1}{3} - \frac{-1}{2} + \frac{-1}{9} - \frac{-1}{6} = -\frac{1}{3} + \frac{1}{2} - \frac{1}{9} + \frac{1}{6} = -\frac{6}{18} + \frac{9}{18} - \frac{2}{18} + \frac{3}{18} \\
&= \frac{-6+9-2+3}{18} = \frac{4}{18} = \frac{2}{9} \approx 0.2222
\end{aligned}
$$

With these probabilities in hand, we can compute the desired conditional probabilities:

$$
\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{9}}{\frac{7}{18}} = \frac{2}{9} \cdot \frac{18}{7} = \frac{4}{7} \approx 0.5714 \\
P(B|A) &= \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{9}}{\frac{3}{4}} = \frac{2}{9} \cdot \frac{4}{3} = \frac{8}{27} \approx 0.2963 \quad \square
\end{aligned}
$$