Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2023 (S62)

Solutions to Quiz #3
Conditional Probability and Independence

Instructions: Do any *two* (2) of the following three problems. (If you do all three, only the first two encountered by the marker will get marked.) Please show all your work.

1. Suppose S is a sample space and A, B, and C are events in S, with $P(A) > 0$, $P(B) > 0$, and $P(C) > 0$, such that $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$. Does it follow that A and B are independent? *[2.5]*

NOTE: You should either explain – correctly! – why A and B must be independent, or give an example of a sample space S and events A, B, and C which meet the above conditions but for which A and B are dependent.

SOLUTION. A and B could be dependent. For example, let $S = \{a, b, c, d, e, f, g, h\}$, with each of the eight outcomes being equally probable, so $P(\omega) = \frac{1}{8} = 0.125$ for each $\omega \in S$. Let $A = \{a, b, c, d\}$, $B = \{d, e, f, g\}$, and $C = \{b, d, f, h\}$, so $A \cap B = \{d\}$, $A \cap C = \{b, d\}$, $B \cap C = \{d, f\}$, and $A \cap B \cap C = \{d\}$. Then $P(A) = P(B) = P(C) = \frac{4}{8} = \frac{1}{2} = 0.5 > 0$, $P(A \cap B) = P(A \cap B \cap C) = \frac{1}{8} = 0.125$, and $P(A \cap C) = P(B \cap C) = \frac{2}{8} = \frac{1}{4} = 0.25$. It follows that $P(A \cap B \cap C) = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B) \cdot P(C)$, as required. However, since $P(A \cap B) = \frac{1}{8} \neq \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B)$, the events A and B are dependent. $\Box$

2. You are given 65 coins, 64 of which are fair and 1 of which has two heads. One coin is selected at random from the 65, and then tossed 6 times, coming up heads on every toss. What is the probability that the selected coin is the two-headed one? *[2.5]*

SOLUTION. This is a job for Bayes' Theorem.

Let C be the event that one of the 64 fair coins is selected, and let D be the event that the two-headed coin is selected. Since one or the other must happen in the process, and since C and D obviously have no outcomes in common, C and D partition the sample space. It is not hard to see that $P(C) = \frac{64}{65}$ and $P(D) = \frac{1}{65}$.

Let E be the event that the selected coin comes up heads every time in six tosses. If the coin is fair, this occurs with a probability of $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$, so $P(E|C) = \frac{1}{64}$. On the other hand, if the coin is two-headed it will come up heads every time in six (or however many) tosses with a probability of 1, so $P(E|D) = 1$.

By Bayes' Theorem, it follows that the probability that the selected coin is the two-headed one, given that the selected coin came up heads every time in six tosses, is:

$$\begin{aligned} P(D|E) &= \frac{P(E|D)P(D)}{P(E|C)P(C) + P(E|D)P(D)} = \frac{1 \cdot \frac{1}{65}}{\frac{1}{64} \cdot \frac{64}{65} + 1 \cdot \frac{1}{65}} \\ &= \frac{\frac{1}{65}}{\frac{1}{65} + \frac{1}{65}} = \frac{\frac{1}{65}}{\frac{2}{65}} = \frac{1}{65} \cdot \frac{65}{2} = \frac{1}{2} = 0.5 \quad \Box \end{aligned}$$

3. A box contains three different types of disposable flashlights. Suppose that 20% of the flashlights in the box are of type A, 30% are of type B, and 50% are of type C. The probabilities that type A, type B, and type C flashlights will last over 100 hours of use are respectively 0.70, 0.40, and 0.30.

a. What is the probability that a flashlight randomly chosen from the box will last over 100 hours of use? *[1]*

b. If a flashlight randomly chosen from the box lasted over 100 hours, what is the probability it was of type B? *[1.5]*

SOLUTIONS. In what follows, let U be the event that the randomly chosen flashlight will last over 100 hours, and let A, B, and C be the events that the randomly chosen flashlight is of the corresponding type. The given information then boils down to $P(A) = 0.2$, $P(B) = 0.3$, $P(C) = 0.5$, $P(U|A) = 0.7$, $P(U|B) = 0.4$, and $P(U|C) = 0.3$.

a. Since the three types of flashlights partition the flashlights in the box, the Rule of Total Probability tells us that the probability of the randomly chosen flashlight lasting over 100 hours, *i.e.* the probability of U, is:

$$\begin{aligned} P(U) &= P(U|A)P(A) + P(U|B)P(B) + P(U|C)P(C) \\ &= 0.7 \cdot 0.2 + 0.4 \cdot 0.3 + 0.3 \cdot 0.5 = 0.14 + 0.12 + 0.15 = 0.41 \quad \square \end{aligned}$$

b. We need to compute the probability that the randomly chosen flashlight is of type B, given that it lasted over 100 hours, *i.e.* the probability of $P(B|U)$. We do so using Bayes' Theorem, with a bit of help from the work in part **a** above:

$$\begin{aligned} P(B|U) &= \frac{P(U|B)P(B)}{P(U|A)P(A) + P(U|B)P(B) + P(U|C)P(C)} = \frac{P(U|B)P(B)}{P(U)} \\ &= \frac{0.4 \cdot 0.3}{0.41} = \frac{0.12}{0.41} = \frac{12}{41} \approx 0.2927 \quad \square \end{aligned}$$