Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2023 (S62)

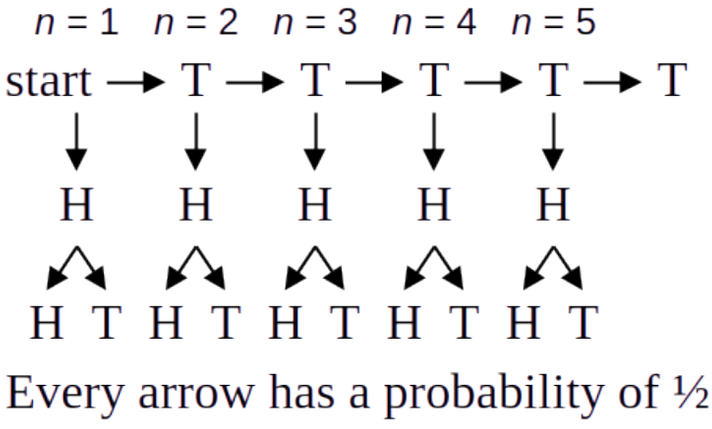
Quiz #2
Tossing

Consider the following experiment:

0. Set the counter n to 1.
1. Toss a fair coin and record the result. Then,
   - if it comes up heads, toss the coin once more and record the result, then end the experiment;
   - if it comes up tails and $n \leq 4$, add 1 to n and repeat step 1;
   - if it comes up tails and $n = 5$, end the experiment.

1. Draw the complete tree diagram for this experiment. *[1]*

SOLUTION. Here it is, with an n-count included:



Every arrow has a probability of ½. □

2. What are the sample space and probability function of this experiment? *[1]*

SOLUTION. The sample space consists of all the sequences of tosses that could occur in this experiment:

$$S = \{\, HH,\ HT,\ THH,\ THT,\ TTHH,\ TTHT,\ TTTHH,\ TTTHT,\ TTTTHH,\ TTTTHT,\ TTTTT \,\}$$

Since the coin is fair, the probability of each outcome is $\left(\frac{1}{2}\right)^X$, where X is the number of tosses required to get that outcome. (One can also think of X as the length of the

outcome.) Explicitly:

$$P(HH) = P(HT) = \frac{1}{4}$$
$$P(THH) = P(THT) = \frac{1}{8}$$
$$P(TTHH) = P(TTHT) = \frac{1}{16}$$
$$P(TTTHH) = P(TTTHT) = P(TTTTT) = \frac{1}{32}$$
$$P(TTTTHH) = P(TTTTHT) = \frac{1}{64} \qquad \square$$

NOTE: The use of the random variable X (defined just before question **4**) above demonstrates one of the applications of random variables: they often make for useful shortcuts.

3. What is the probability that ...

a. the final toss is a tail? *[0.5]*

b. the next-to-last toss is a head? *[0.5]*

SOLUTIONS. **a.** Here we go:

$$\begin{aligned} P(\text{final toss is a tail}) &= P(HT) + P(THT) + P(TTHT) + P(TTTHT) \\ &\quad + P(TTTTT) + P(TTTTHT) \\ &= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32} + \frac{1}{64} \\ &= \frac{16}{64} + \frac{8}{64} + \frac{4}{64} + \frac{2}{64} + \frac{2}{64} + \frac{1}{64} \\ &= \frac{33}{64} = 0.015625 \qquad \square \end{aligned}$$

b. Looking at the tree, note that the only outcome in which the next-to-last toss is not a head, is $TTTTT$. Thus:

$$\begin{aligned} P(\text{next-to-last toss is a head}) &= 1 - P(\text{next-to-last toss is a tail}) \\ &= 1 - \frac{1}{32} = \frac{31}{32} = 0.03125 \qquad \square \end{aligned}$$

The random variable X counts the number of tosses made in this experiment.

4. What are the possible values of X and the probabilities that each will occur? *[1]*

SOLUTION. Looking at the outcomes in the sample space S, one can see that X take on the values 2, 3, 4, 5, or 6. Their probabilities are:

$$P(X=2) = P(HH) + P(HT) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$P(X=3) = P(THH) + P(THT) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$
$$P(X=4) = P(TTHH) + P(TTHT) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$$
$$P(X=5) = P(TTTHH) + P(TTTHT) + P(TTTTT) = \frac{1}{32} + \frac{1}{32} + \frac{1}{32} = \frac{3}{32}$$
$$P(X=6) = P(TTTTHH) + P(TTTTHT) = \frac{1}{64} + \frac{1}{64} = \frac{1}{32} \qquad \square$$

NOTE: As long as all you want to know is captured by X, you could assume this is some process that has sample space $T = \{\, 2,\ 3,\ 4,\ 5,\ 6 \,\}$, with probabilities as above.

5. What is the probability that ...

a. X is an even number? *[0.5]*

b. X is at least four? *[0.5]*

SOLUTIONS. **a.** Here we go:

$$\begin{aligned} P(X \text{ is even}) &= P(X=2) + P(X=4) + P(X=6) \\ &= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} = \frac{16}{32} + \frac{4}{32} + \frac{1}{32} = \frac{21}{32} = 0.65625 \qquad \square \end{aligned}$$

b. *Direct approach.*

$$\begin{aligned} P(X \geq 4) &= P(X=4) + P(X=5) + P(X=6) \\ &= \frac{1}{8} + \frac{3}{32} + \frac{1}{32} = \frac{4}{32} + \frac{3}{32} + \frac{1}{32} = \frac{8}{32} = \frac{1}{4} = 0.25 \qquad \square \end{aligned}$$

b. *Slightly indirect approach.*

$$\begin{aligned} P(X \geq 4) &= 1 - P(X < 4) = 1 - [P(X=2) + P(X=3)] \\ &= 1 - \left[\frac{1}{2} + \frac{1}{4}\right] = 1 - \left[\frac{2}{4} + \frac{1}{4}\right] = 1 - \frac{3}{4} = \frac{1}{4} = 0.25 \qquad \square \end{aligned}$$