Mathematics 1550H – Probability I: Introduction to Probability

Trent University, Summer 2023 (S62)

Solution to Quiz #0
Urning Your Way

You are given two large urns, identical in every way, and 73 tennis balls, identical except for their colour: 29 are green, 23 are blue, and the rest are red. It will be your task to put all the tennis balls into the two urns, with the only restrictions being that you must indeed put each and every tennis ball into one urn or the other, and that each urn must end up with at least one tennis ball. Once you have completed your task, a blindfolded person will be brought in and asked to choose one of the two urns and then select a tennis ball from the chosen urn.

1. How should you distribute the 73 tennis balls to maximize the chance that a green ball will be selected? Explain why your method does the job. *[5]*

Solution. The optimal solution is to put 1 green ball in one of the urns and all the 72 other balls in the other. If the urn with the one green ball is chosen, the probability after that of selecting a green ball is 1; if the urn with 72 balls is chosen, the probability of selecting a green ball after that is $\dfrac{29-1}{72} = \dfrac{28}{72} = \dfrac{7}{18}$. Given that the two urns are equally likely to be chosen to select a ball from, this will make the overall probability of a green ball being selected in this process $\dfrac{1}{2} \cdot 1 + \dfrac{1}{2} \cdot \dfrac{7}{18} = \dfrac{18}{36} + \dfrac{7}{36} = \dfrac{25}{36} \approx 0.6944$.

Why is this solution optimal? The easiest way to see it is to experiment a little. If we take a few green balls out of the urn with lots of balls and put them in the other, we don't increase the probability of 1 that a green ball would be selected from that urn, but we do decrease the probability that one would be selected from the urn they were taken from. Similarly, if we take a few non-green balls from the urn with many balls to put in the other, we drastically decrease the probability that a green ball would be selected from that urn, and only slightly increase the probability of selecting one from the urn with many balls. So our solution is at least some sort of local maximum.

For example, suppose we put 15 green and 20 non=green balls in one urn and the remaining 14 green and 24 non-green balls in the other. The probability that the given process will select a green ball is then $\frac{1}{2} \cdot \frac{15}{15+20} + \frac{1}{2} \cdot \frac{14}{14+24} = \frac{12}{\cdot}\frac{3}{7} + \frac{1}{2} \cdot \frac{7}{17} = \frac{50}{119} \approx 0.4202$, which is rather less than for the optimal solution.

A fully general argument would require mathematical additional tools, such as multivariate calculus, which are probably beyond the scope of this course. If we put g green and n non-green balls in one urn, and hence put $29-g$ green and $44-n$ non-green balls in the other urn, the probability that a green ball will end up being selected will be $\dfrac{1}{2} \cdot \dfrac{g}{g+n} + \dfrac{1}{2} \cdot \dfrac{29-g}{73-g-n}$. One would need to maximize this expression subject to the constraints that $0 \le g \le 29$, $0 \le n \le 44$, and also $g+n \ge 1$ and $73-g-n \ge 1$, since neither urn is allowed to be empty. (And we're not allowed to divide by 0 ...) $\square$