Mathematics 1550H – Probability I: Introduction to Probability
Trent University, Summer 2023 (S62)

Final Examination
14:00-14:50 Tuesday, 1 August, in ENW 117

Instructions: Do both of parts **A** and **B**, and, if you wish, part **C**. Show all your work and simplify answers as much as practical. *If in doubt about something,* **ask!**

Aids: Calculator, letter- or A4-size aid sheet, standard normal table (supplied), one brain.

Part A. Do all of **1**–**5**. *[Subtotal = 60/90]*

1. A fair coin is tossed twice. If it comes up heads on the second toss, the experiment ends; if it comes up tails, the coin is tossed twice more and then the experiments ends.

- **a.** Draw the complete tree diagram for this experiment. *[3]*
- **b.** What are the sample space and probability function for this experiment? *[5]*
- **c.** Let E be the event that an even number of heads comes up in the course of the experiment and let F be the event that four tosses are made during the experiment. Determine whether the events E and F are independent or not. *[5]*

2. Suppose the continuous random variable X has $f(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ as its probability density function.

- **a.** Check that $f(x)$ is a valid probability density function. *[7]*
- **b.** Compute the expected value $E(X)$ and the variance $V(X)$ of X. *[8]*

3. A five-card hand drawn from a standard 52-card deck is a *straight* if it is made of cards of consecutive kinds,* where the order of the kinds wraps around the ends. For example, the hand 4♠, 3♣, 2♣, A♡, K♠ would count as a straight.

- **a.** If the order in which the hand is drawn doesn't matter, how many different straights are possible? *[5]*
- **b.** What is the probability that a hand is a straight, given that it is a *flush*, *i.e.* that all the cards in the hand are from the same suit? *[7]*

4. Suppose W is continuous random variable with a normal distribution that has $\mu = E(W) = 3$ and $\sigma^2 = V(W) = 4$. Compute $P(W \geq 1.7 \mid W \leq 3.9)$ with the help of a standard normal table. *[10]*

5. Die A is a fair standard die and die B is fair four-sided die with faces numbered 1 through 4. One of the two dice is chosen at random, with equal likelihood, and rolled. The random variable X records the number on the face of the die that came up on the roll. Use Bayes' Theorem to compute the probability that die B had been chosen, given that $X = 2$. *[10]*

*[Parts **B** and **C** are on page 2.]*

* The *kinds* are, in order, A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, and 2, and the *suits* are ♡, ◇, ♣, and ♠.

*[Part **A** is on page 1.]*

Part B. Do any *two* (2) of **6–9**. *[Subtotal = 30/90]*

6. An urn contains 10 balls, 9 of which are purple and 1 of which is green. The balls are drawn from the urn, one at a time and without replacement, until the green ball is drawn. The random variable X tells us which draw the green ball appeared on.

a. What is the probability distribution function of X? *[6]*

b. What kind of distribution does X have? *[1]*

c. What are the expected value $E(X)$ and variance $V(X)$ of X? *[8]*

7. The continuous random variable X has $g(x) = \begin{cases} xe^{-x^2} & x \geq 0 \\ -xe^{-x^2} & x < 0 \end{cases}$ as its probability density function.

a. Verify that $g(x)$ is a valid probability density function. *[6]*

b. Without doing any calculus, what is the expected value $E(X)$ of X? *[2]*

c. With at least some calculus, what is the variance $V(X)$ of X? *[7]*

8. Suppose Y is a continuous random variable that has a normal distribution with expected value $E(Y) = 10$ and variance $V(X) = 9$.

a. Find an upper bound for $P(Y \geq 20)$ using Markov's Inequality. *[3]*

b. Find an upper bound for $P(Y \geq 20)$ using Chebyshev's Inequality. *[6]*

b. Compute $P(Y \geq 20)$ with the help of a standard normal table. *[6]*

9. Suppose the discrete random variables X and Y are jointly distributed according to the given table.

$Y \backslash X$	0	1	2
1	0	0.2	0.2
2	0.3	0	0.1
3	0.1	0.1	0

a. Compute the expected values $E(X)$ and $E(Y)$, the variances $V(X)$ and $V(Y)$, and also the covariance $\text{Cov}(X, Y)$ of X and Y. *[10]*

b. Determine whether X and Y are independent. *[1]*

c. Let $U = X + 3Y$. Compute $E(U)$ and $V(U)$. *[4]*

[Total = 90]

Part C. *Bonus time!*

T. Suppose X_1 and X_2 each have an exponential distribution with $\lambda = 1$. What kind of distribution does $X = X_1 + X_2$ have? (No calculation or proof required, just an answer.) *[1]*

H. Write a haiku touching on probability or mathematics in general. *[1]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

I HOPE THAT YOU ENJOYED THE COURSE. ENJOY THE REST OF THE SUMMER!