Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2023 (S62)

SOLUTIONS TO THE FINAL EXAMINATION
14:00-14:50 Tuesday, 1 August, in ENW 117

Instructions: Do both of parts **A** and **B**, and, if you wish, part **C**. Show all your work and simplify answers as much as practical. *If in doubt about something,* **ask!**

Aids: Calculator, letter- or A4-size aid sheet, standard normal table (supplied), one brain.

Part A. Do all of **1**–**5**. *[Subtotal = 60/90]*

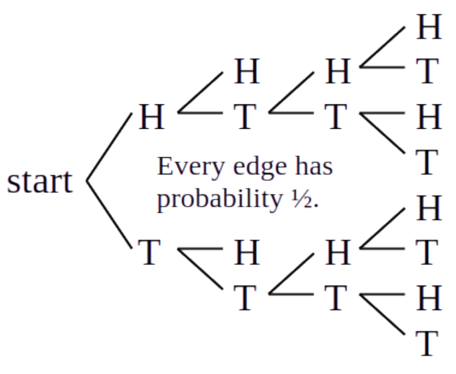
1. A fair coin is tossed twice. If it comes up heads on the second toss, the experiment ends; if it comes up tails, the coin is tossed twice more and then the experiments ends.

a. Draw the complete tree diagram for this experiment. *[3]*

b. What are the sample space and probability function for this experiment? *[5]*

c. Let E be the event that an even number of heads comes up in the course of the experiment and let F be the event that four tosses are made during the experiment. Determine whether the events E and F are independent or not. *[5]*

SOLUTION. **a.** Here it is:



□

b. The samples space is

$$S = \{ HH, TH, HTHH, HTHT, HTTH, HTTT, TTHH, TTHT, TTTH, TTTT \},$$

and the probability distribution function, since the coin is fair, is given by $m(HH) = m(TT) = \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 0.25$ and $m(\omega) = \left(\frac{1}{2}\right)^4 = \frac{1}{16} = 0.0625$ if $\omega \in S \setminus \{ HH, TH \}$. □

c. We have $E = \{ HH, HTHT, HTTH, TTHH, TTTT \}$, so $P(E) = \frac{1}{4} + \frac{4}{16} = \frac{1}{2} = 0.5$, $F = \{ HTHH, HTHT, HTTH, HTTT, TTHH, TTHT, TTTH, TTTT \}$, so $P(F) = \frac{8}{16} = \frac{1}{2} = 0.5$, and $E \cap F = \{ HTHT, HTTH, TTHH, TTTT \}$, so $P(E \cap F) = \frac{4}{16} = \frac{1}{4} = 0.25$. Since $P(E \cap F) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(E)P(F)$, it follows by definition that E and F are independent. ■

2. Suppose the continuous random variable X has $f(x) = \begin{cases} 3x^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$ as its probability density function.

a. Check that $f(x)$ is a valid probability density function. *[7]*

b. Compute the expected value $E(X)$ and the variance $V(X)$ of X. *[8]*

SOLUTION. **a.** Observe that $f(x) = 3x^2 \geq 0$ (since $x^2 \geq 0$) when $0 \leq x \leq 1$, and $f(x)0 \geq 0$ otherwise. Thus $f(x) \geq 0$ for all x, meeting the first requirement for a valid probability density function.

For the second requirement for a valid probability density function we need to check that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Here we go:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 3x^2\,dx + \int_{1}^{\infty} 0\,dx = 0 + 3\frac{x^3}{3}\bigg|_0^1 + 0 \\ &= x^3\Big|_0^1 = 1^3 - 0^3 = 1 - 0 = 1 \end{aligned}$$

Since it satisfies both requirements to be a valid probability density function, $f(x)$ is one. □

b. We set up the relevant definitions and compute away:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{1} x \cdot 3x^2\,dx + \int_{1}^{\infty} x \cdot 0\,dx \\ &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 3x^3\,dx + \int_{1}^{\infty} 0\,dx = 0 + 3\frac{x^4}{4}\bigg|_0^1 + 0 \\ &= \frac{3}{4}1^4 - \frac{3}{4}0^4 = \frac{3}{4} \cdot 1 - \frac{3}{4} \cdot 0 = \frac{3}{4} - 0 = \frac{3}{4} = 0.75 \\ E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{0} x^2 \cdot 0\,dx + \int_{0}^{1} x^2 \cdot 3x^2\,dx + \int_{1}^{\infty} x^2 \cdot 0\,dx \\ &= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 3x^5\,dx + \int_{1}^{\infty} 0\,dx = 0 + 3\frac{x^5}{5}\bigg|_0^1 + 0 \\ &= \frac{3}{5}1^5 - \frac{3}{5}0^5 = \frac{3}{5} \cdot 1 - \frac{3}{5} \cdot 0 = \frac{3}{5} - 0 = \frac{3}{5} = 0.6 \\ V(X) &= E\left(X^2\right) - [E(X)]^2 = \frac{3}{5} - \left[\frac{3}{4}\right]^2 = \frac{3}{5} - \frac{9}{16} \\ &= \frac{48}{80} - \frac{45}{80} = \frac{3}{80} = 0.0375 \qquad \blacksquare \end{aligned}$$

3. A five-card hand drawn from a standard 52-card deck is a *straight* if it is made of cards of consecutive kinds,* where the order of the kinds wraps around the ends. For example, the hand $4\spadesuit$, $3\clubsuit$, $2\clubsuit$, $A\heartsuit$, $K\spadesuit$ would count as a straight.

a. If the order in which the hand is drawn doesn't matter, how many different straights are possible? *[5]*

b. What is the probability that a hand is a straight, given that it is a *flush*, *i.e.* that all the cards in the hand are from the same suit? *[7]*

SOLUTION. **a.** If only the kinds are considered, there are 13 possible straights because here are 13 kinds We can list them by the "top" card: $A\,K\,Q\,J\,10$, $K\,Q\,J\,10\,9$, ... , $2\,A\,K\,Q\,J$. However, each of the five cards in a straight could come from any of the four suits. There are therefore $13 \cdot 4^5 = 13312$ possible straights. □

b. To get a flush, we need to pick one of the 4 suits and then choose a hand of five cards from the 13 cards in that suit. There are therefore $4 \cdot \binom{13}{5} = 4 \cdot 1287 = 5148$ possible flushes. Since each of the $\binom{52}{5} = 2598960$ possible five-card hands (as order doesn't matter) is equally likely, the probability of a hand being a flush is $\frac{5148}{2598960}$.

For a hand to be both a flush and a straight, we need to pick one of the 4 suits and then choose one of the 13 cards in that suit to be the "top" card of the straight, as in **a** above. There are therefore $4 \cdot 13 = 52$ possible hands that are both a straight and a flush. Since each of the $\binom{52}{5} = 2598960$ possible five-card hands is equally likely, the probability of a hand being a straight and a flush is $\frac{52}{2598960}$.

It now follows that the probability that a hand is a straight, given that it is a flush, is:

$$P(\text{straight}|\text{flush}) = \frac{P(\text{straight and flush})}{P(\text{flush})} = \frac{\frac{52}{2598960}}{\frac{5148}{2598960}} = \frac{52}{5148} = \frac{1}{99} \approx 0.0101 \quad \blacksquare$$

* The *kinds* are, in order, A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, and 2, and the *suits* are $\heartsuit$, $\diamondsuit$, $\clubsuit$, and $\spadesuit$.

4. Suppose W is continuous random variable with a normal distribution that has $\mu = E(W) = 3$ and $\sigma^2 = V(W) = 4$. Compute $P(W \geq 1.7 \mid W \leq 3.9)$ with the help of a standard normal table. *[10]*

SOLUTION. We convert the necessary probabilities for W into ones for the standard normal random variable $Z = \frac{W-\mu}{\sigma}$. Note that the standard deviation of W is $\sigma = \sqrt{V(W)} = \sqrt{4} = 2$. Here we go, with the help of our standard normal table:

$$\begin{aligned}
P(W \leq 3.9) &= P\left(\frac{W-3}{2} \leq \frac{3.9-3}{2}\right) = P(Z \leq 0.45) \approx 0.6736 \\
P(W \geq 1.7 \text{ and } W \leq 3.9) &= P(1.7 \leq W \leq 3.9) = P\left(\frac{1.7-3}{2} \leq \frac{W-3}{2} \leq \frac{3.9-3}{2}\right) \\
&= P(-0.65 \leq Z \leq 0.45) = P(Z \leq 0.45) - P(Z < -0.65) \\
&\approx 0.6736 - 0.2578 = 0.4158 \\
P(W \geq 1.7 \mid W \leq 3.9) &= \frac{P(W \geq 1.7 \text{ and } W \leq 3.9)}{P(W \leq 3.9)} \approx \frac{0.4158}{0.6736} \approx 0.6173 \qquad \blacksquare
\end{aligned}$$

5. Die A is a fair standard die and die B is fair four-sided die with faces numbered 1 through 4. One of the two dice is chosen at random, with equal likelihood, and rolled. The random variable X records the number on the face of the die that came up on the roll. Use Bayes' Theorem to compute the probability that die B had been chosen, given that $X = 2$. *[10]*

SOLUTION. The probability that $X = 2$, *i.e* 2 came up when the chosen die was rolled, if die A was the one chosen is $P(X = 2 \mid A) = \frac{1}{6}$, since 2 is on just one of the six faces of die A. Similarly, the probability that $X = 2$ if die B was chosen is $P(X = 2 \mid B) = \frac{1}{4}$. We are given that the dice are equally likely to be chosen, *i.e.* that $P(A) = P(B) = \frac{1}{2}$. By Bayes' Theorem, it now follows that the probability that die B was chosen, given that $X = 2$, is:

$$\begin{aligned}
P(B \mid X = 2) &= \frac{P(X = 2 \mid B)P(B)}{P(X = 2 \mid A)P(A) + P(X = 2 \mid B)P(B)} = \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{6} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2}} = \frac{\frac{1}{8}}{\frac{1}{12} + \frac{1}{8}} \\
&= \frac{\frac{1}{8}}{\frac{2}{24} + \frac{3}{24}} = \frac{\frac{1}{8}}{\frac{5}{24}} = \frac{1}{8} \cdot \frac{24}{5} = \frac{3}{5} = 0.6 \qquad \blacksquare
\end{aligned}$$

Part B. Do any *two* (2) of **6–9**. *[Subtotal = 30/90]*

6. An urn contains 10 balls, 9 of which are purple and 1 of which is green. The balls are drawn from the urn, one at a time and without replacement, until the green ball is drawn. The random variable X tells us which draw the green ball appeared on.

- **a.** What is the probability distribution function of X? *[6]*
- **b.** What kind of distribution does X have? *[1]*
- **c.** What are the expected value $E(X)$ and variance $V(X)$ of X? *[8]*

Solution. **a.** Note that X can take on the integer values 1 though 10; since balls are not replaced and there are 10 balls, only one of which is green, there can be at most 10 draws. On any given draw, if there are a purple balls and the green ball remaining, one has a probability of $\frac{a}{a+1}$ of drawing a purple ball and a probability of $\frac{1}{a}$ of drawing the green one. The probability distribution function $m(k) = P(X = k)$ of X is then given by:

$$
\begin{aligned}
m(1) = P(X = 1) &= \frac{1}{10} \\
m(2) = P(X = 2) &= \frac{9}{10} \cdot \frac{1}{9} = \frac{1}{10} \\
m(3) = P(X = 3) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} = \frac{1}{10} \\
m(4) = P(X = 4) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{1}{7} = \frac{1}{10} \\
m(5) = P(X = 5) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} \cdot \frac{1}{6} = \frac{1}{10} \\
m(6) = P(X = 6) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} \cdot \frac{5}{6} \cdot \frac{1}{5} = \frac{1}{10} \\
m(7) = P(X = 7) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{10} \\
m(8) = P(X = 8) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{10} \\
m(9) = P(X = 9) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{10} \\
m(10) = P(X = 10) &= \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} \cdot \frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{10} \qquad \square
\end{aligned}
$$

b. Since every value of X is equally likely, X has a uniform distribution. $\square$

c. Here we go:

$$
\begin{aligned}
E(X) &= \sum_{k=1}^{10} kP(X = k) = \sum_{k=1}^{10} k \cdot \frac{1}{10} = \frac{1}{10} \sum_{k=1}^{10} k = \frac{1}{10} \cdot \frac{10(10+1)}{2} = \frac{11}{2} = 5.5 \\
E\left(X^2\right) &= \sum_{k=1}^{10} k^2 P(X = k) = \sum_{k=1}^{10} k^2 \cdot \frac{1}{10} = \frac{1}{10} \sum_{k=1}^{10} k^2 = \frac{1}{10} \cdot \frac{10(10+1)(2 \cdot 10 + 1)}{6} \\
&= \frac{231}{6} = 38.5 \\
V(X) &= E\left(X^2\right) - [E(X)]^2 = 38.5 - 5.5^2 = 38.5 - 30.25 = 7.25 \qquad \blacksquare
\end{aligned}
$$

7. The continuous random variable X has $g(x) = \begin{cases} xe^{-x^2} & x \geq 0 \\ -xe^{-x^2} & x < 0 \end{cases}$ as its probability density function.

a. Verify that $g(x)$ is a valid probability density function. *[6]*

b. Without doing any calculus, what is the expected value $E(X)$ of X? *[2]*

c. With at least some calculus, what is the variance $V(X)$ of X? *[7]*

SOLUTION. **a.** Since $e^{-x^2} > 0$ for all x, $g(x) = xe^{-x^2} \geq 0$ when $x \leq 0$ and, since it is also true that $-x > 0$ when $x < 0$, $g(x) = -xe^{-x^2} > 0$ when $x < 0$. Thus $g(x) \geq 0$ for all x, satisfying the first condition needed to be a valid probability density.

For the second condition needed to be a valid probability density, we need to check that $\int_{-\infty}^{\infty} g(x)\,dx = 1$. We will do so with the help of the substitution $u = -x^2$, so $du = -2x\,dx$, and hence $x\,dx = \left(-\frac{1}{2}\right) du$. We will also change the limits as we go along: $\begin{array}{cccc} x & -\infty & 0 & \infty \\ u & -\infty & 0 & -\infty \end{array}$. Here goes:

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)\,dx &= \int_{-\infty}^{0} (-1)xe^{-x^2}\,dx + \int_{0}^{\infty} xe^{-x^2}\,dx \\
&= \int_{-\infty}^{0} (-1)e^{u}\left(-\frac{1}{2}\right) du + \int_{0}^{-\infty} e^{u}\left(-\frac{1}{2}\right) du \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{u}\,du - \frac{1}{2}\int_{0}^{-\infty} e^{u}\,du \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{u}\,du + \frac{1}{2}\int_{-\infty}^{0} e^{u}\,du = \int_{-\infty}^{0} e^{u}\,du \\
&= e^{u}\big|_{-\infty}^{0} = e^{0} - e^{-\infty} = 1 - 0 = 1
\end{aligned}$$

Thus $g(x)$ meets both conditions needed to be a valid probability density function. □

b. Observe that $g(x)$ is an even function, that is, $g(-x) = g(x)$ for all x: if $x < 0$, then $-x > 0$, so $g(-x) = (-x)e^{-(-x)^2} = -xe^{-x^2} = g(x)$; if $x > 0$, then $-x < 0$, so $g(-x) = -(-x)e^{-(-x)^2} = xe^{-x^2} = g(x)$; and if $x = 0$, then $-0 = 0 \geq 0$, so $g(-0) = g(0) = 0e^{-0^2} = 0$. It follows that $xg(x)$ is an odd function, that is, $(-x)g(-x) = -xg(x)$ for all x. This , in turn means that $\displaystyle\int_{-\infty}^{0} xg(x)\,dx = -\int_{0}^{\infty} xg(x)\,dx$, which is true for any odd function defined for all x for which the intergrals are defined, so $E(X) = \displaystyle\int_{-\infty}^{\infty} xg(x)\,dx = \int_{-\infty}^{0} xg(x)\,dx + \int_{0}^{\infty} xg(x)\,dx = 0$.

A little more informally and without mentioning the integrals (they are mentions of calculus, even if we didn't do any :-), each positive x has the same probability $g(x) = g(-x)$ as its negative x, so the weighted probabilities cancel out, leaving 0 as the only possibility for the expected value of X. □

c. Our principal tool will be the same substitution used in part **a**, namely $u = -x^2$, so $x^2 = -u$, $du = -2x\,dx$, and hence $x\,dx = \left(-\frac{1}{2}\right)\,du$, with the limits changing accordingly, $\begin{array}{cccc} x & -\infty & 0 & \infty \\ u & -\infty & 0 & -\infty \end{array}$. Here goes:

$$\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 g(x)\,dx = \int_{-\infty}^{0} x^2 \cdot (-1)xe^{-x^2}\,dx + \int_{0}^{\infty} x^2 \cdot xe^{-x^2}\,dx \\
&= \int_{-\infty}^{0} (-1)x^3e^{-x^2}\,dx + \int_{0}^{\infty} x^3e^{-x^2}\,dx \\
&= \int_{-\infty}^{0} ue^u\left(-\frac{1}{2}\right)\,du + \int_{0}^{-\infty} (-u)e^u\left(-\frac{1}{2}\right)\,du \\
&= -\frac{1}{2}\int_{-\infty}^{0} ue^u\,du + \frac{1}{2}\int_{0}^{-\infty} ue^u\,du = \frac{1}{2}\int_{-\infty}^{0} ue^u\,du - \frac{1}{2}\int_{-\infty}^{0} ue^u\,du \\
&= -\int_{-\infty}^{0} ue^u\,du \qquad \text{Use integration by parts with } u = u \text{ and } v' = e^u\text{, so } u' = 1 \text{ and } v = e^u. \\
&= -\left[ue^u\big|_{-\infty}^{0} - \int_{-\infty}^{0} 1e^u\,du\right] = -\left[0e^0 - (-\infty)e^{-\infty}\right] + e^u\big|_{-\infty}^{0} \\
&= -[0-0] + \left[e^0 - e^{-\infty}\right] = 0 + [1-0] = 1
\end{aligned}$$

We were a little informal with evaluating the expressions at the end, exploiting the fact that exponential dominate polynomials, so, *e.g.* $\infty e^{-\infty} = \frac{\infty}{e^{\infty}} = 0$. We can now compute the variance of X:

$$V(X) = E\left(X^2\right) - [E(X)]^2 = 1 - 0^2 = 1 \quad \blacksquare$$

8. Suppose Y is a continuous random variable that has a normal distribution with expected value $E(Y) = 10$ and variance $V(X) = 9$.

a. Find an upper bound for $P(Y \geq 20)$ using Markov's Inequality. *[3]*

b. Find an upper bound for $P(Y \geq 20)$ using Chebyshev's Inequality. *[6]*

c. Compute $P(Y \geq 20)$ with the help of a standard normal table. *[6]*

SOLUTION. **a.** Since every normal distribution takes on some negative values and Markov's Inequality only applies to non-negative random variables, we can't legitimately apply Markov's Inequality in this case. If we blindly tried anyway, we would "get" something like $P(Y \geq 20) \leq \frac{E(Y)}{20} = \frac{10}{20} = \frac{1}{2} = 0.5$. $\square$

b. *A little crude.* Recall that X has expected value $E(X) = 10$ and note that X has standard deviation $\sigma = \sqrt{V(X)} = \sqrt{9} = 3$. Observe that $P(Y \geq 20) = P(Y - 10 \geq 20 - 10) = P(Y - 10 \geq 10) \leq P(|Y - 10| \geq 10)$, since $|Y - 10| \geq 10$ captures more values of Y than $Y - 10 \geq 10$. We can apply Chebyshev's Inequality to $(|Y - 10| \geq 10)$ if we decompose 10 into $k\sigma = 3k$: if $10 = 3k$, then $k = \frac{10}{3}$. It now follows that:

$$
\begin{aligned}
P(Y \geq 20) \leq P(|Y - 10| \geq 10) &= P\left(|Y - 10| \geq \frac{10}{3} \cdot 3\right) \\
&\leq \frac{1}{\left(\frac{10}{3}\right)^2} = \left(\frac{3}{10}\right)^2 = \frac{9}{100} = 0.09 \qquad \square
\end{aligned}
$$

b. *A little clever.* We'll use all the information in the slightly crude approach above, with one extra bit thrown in. A normal distribution's density function is symmetric about the mean of the distribution which in this case is $E(Y) = 10$. It follows that $P(Y - 10 \geq 10) = P(Y - 10 \leq -10)$ and, since $P(|Y - 10| \geq 10) = P(Y - 10 \leq -10) + P(Y - 10 \geq 10) = 2\,P(Y - 10 \geq 10)$, we have that $P(Y - 10 \geq 10) = \frac{1}{2}\,P(|Y - 10| \geq 10)$. Reusing the calculation in the crude solution, it now follows that

$$
P(Y \geq 20) = P(Y - 10 \geq 10) = \frac{1}{2}\,P(|Y - 10| \geq 10) \leq \frac{1}{2} \cdot 0.09 = 0.045\,,
$$

which cuts the upper bound obtained in the slightly crude solution in half. $\square$

c. Since Y has a normal distribution with expected value $\mu = E(Y) = 10$ and variance $\sigma^2 = V(X) = 9$, and hence standard deviation $\sigma = 3$, $Z = \frac{Y - 10}{3}$ has a standard normal distribution. We convert the desired probability into one for Z and evaluate it using the suoolied standard normal table:

$$
\begin{aligned}
P(Y \geq 20) &= P\left(\frac{Y - 10}{3} \geq \frac{20 - 10}{3}\right) = P\left(Z \geq \frac{10}{3}\right) = 1 - P\left(Z < \frac{10}{3}\right) \\
&\approx 1 - P(Z < 3.33) \approx 1 - 0.9996 = 0.0004
\end{aligned}
$$

Note that this is much smaller than either of the upper bounds we obtained in part **b**. ■

9. Suppose the discrete random variables X and Y are jointly distributed according to the given table.

$Y \backslash X$	0	1	2
1	0	0.2	0.2
2	0.3	0	0.1
3	0.1	0.1	0

a. Compute the expected values $E(X)$ and $E(Y)$, the variances $V(X)$ and $V(Y)$, and also the covariance $\operatorname{Cov}(X,Y)$ of X and Y. *[10]*

b. Determine whether X and Y are independent. *[1]*

c. Let $U = X + 3Y$. Compute $E(U)$ and $V(U)$. *[4]*

SOLUTION. **a.** Here we go:

$$\begin{aligned}
E(X) &= 0(0+0.3+0.1)+1(0.2+0+0.1)+2(0.2+0.1+0) \\
&= 0\cdot 0.4+1\cdot 0.3+2\cdot 0.3 = 0+0.3+0.6 = 0.9 \\
E(Y) &= 1(0+0.2+0.2)+2(0.3+0+0.1)+3(0.1+0.1+0) \\
&= 1\cdot 0.4+2\cdot 0.4+3\cdot 0.2 = 0.4+0.8+0.6 = 1.8 \\
E\left(X^2\right) &= 0^2(0+0.3+0.1)+1^2(0.2+0+0.1)+2^2(0.2+0.1+0) \\
&= 0\cdot 0.4+1\cdot 0.3+4\cdot 0.3 = 0+0.3+1.2 = 1.5 \\
E\left(Y^2\right) &= 1^2(0+0.2+0.2)+2^2(0.3+0+0.1)+3^2(0.1+0.1+0) \\
&= 1\cdot 0.4+4\cdot 0.4+9\cdot 0.2 = 0.4+1.6+1.8 = 3.8 \\
V(X) &= E\left(X^2\right)-[E(X)]^2 = 1.5-[0.9]^2 = 1.5-0.81 = 0.69 \\
V(Y) &= E\left(Y^2\right)-[E(Y)]^2 = 3.8-[1.8]^2 = 3.8-3.24 = 0.56 \\
E(XY) &= 0\cdot 1\cdot 0+1\cdot 1\cdot 0.2+2\cdot 1\cdot 0.2 \\
&\quad +0\cdot 2\cdot 0.3+1\cdot 2\cdot 0+2\cdot 2\cdot 0.1 \\
&\quad +0\cdot 3\cdot 0.1+1\cdot 3\cdot 0.1+2\cdot 3\cdot 0 \\
&= 0+0.2+0.4+0+0+0.4+0+0.3+0 = 1.3 \\
\operatorname{Cov}(X,Y) &= E(XY)-E(X)E(Y) = 1.3-0.9\cdot 1.8 = 1.3-1.62 = -0.32 \qquad \square
\end{aligned}$$

b. Since $\operatorname{Cov}(X,Y) = -0.32 \neq 0$, X and Y cannot be independent. $\square$

c. Here we go, using the properties of expected value, variance, and covariance:

$$\begin{aligned}
E(U) &= E(X+3Y) = E(X)+E(3Y) = E(X)+3E(Y) = 0.9+3\cdot 1.8 = 6.3 \\
V(U) &= V(X+3Y) = V(X)+V(3Y)+2\operatorname{Cov}(X,3Y) \\
&= V(X)+3^2V(Y)+2\cdot 3\operatorname{Cov}(X,3Y) \\
&= 0.69+9\cdot 0.56+6\cdot(-0.32) = 0.69+5.04-1.92 = 3.81 \qquad \blacksquare
\end{aligned}$$

[Total = 90]

Part C. *Bonus time!*

T. Suppose X_1 and X_2 each have an exponential distribution with $\lambda = 1$. What kind of distribution does $X = X_1 + X_2$ have? (No calculation or proof required, just an answer.) *[1]*

SOLUTION. The density function of each of X_1 and X_2 is $f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$, from which it follows (see Example 7.4 on pp. 292-293 in the textbook) that the density function of $X = X_1 + X_2$ is $g(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$. Not itself an exponential distribution, but closely related. ■

NOTE: Guessing that $X = X_1 + X_2$ has an exponential distribution earned a half-point.

H. Write a haiku touching on probability or mathematics in general. *[1]*

haiku?

seventeen in three:
five and seven and five of
syllables in lines

I HOPE THAT YOU ENJOYED THE COURSE. ENJOY THE REST OF THE SUMMER!