Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2023 (S62)

Assignment #6
Sadistics – er, Statistics

Suppose we have a coin with a probability p of coming up heads and $q = 1 - p$ of coming up tails on any given toss (so the coin is biased unless $p = 0.5$), but we are not given what p or q are. We know, from class or the textbook, that the expected value of the number of heads in n tosses is $E(X) = np$. If we repeatedly flip the coin and record the results, the number of heads that actually turn up, call it $\hat{E}$, can be divided by the number of tosses, n, to give an estimate, $\hat{p} = \hat{E}/n$, of p. The problem is that we're likely to have to be pretty lucky for $\hat{p} = p$, so the real question is how likely it is that $\hat{p}$ is close to p. (This sort of thing is what statistics is all about: try infer from actual data what is really true, and also provide an estimate of the likelihood that what is inferred is close to reality.)

Let us suppose, for the sake of argument, that we know [somehow!] that $0.4 \leq p \leq 0.6$ (so $0.4 \leq q = 1 - p \leq 0.6$) for our possibly biased coin and that we have the time and patience to toss it any finite number of times n that we need to. Further, let us suppose that we desire to get an estimate of p that is within 5% of the real value, *i.e.* $0.95p \leq \hat{p} \leq 1.05p$.

1. How many times do we need to toss the coin, keeping track of the number of tosses and the number of success, to ensure that $P(0.95p \leq \hat{p} \leq 1.05p) \geq 0.95$? Please justify your answer as fully as you can. *[10]*

NOTE: That is, find an n that ensures that the probability that $\hat{p}$ is within 5% of p is at least 0.95.

SOLUTION. We will take a relatively simple approach using Chebyshev's Inequality that will give us a value of n that is likely to give an n serious overkill. Trying for the least n that does the job is much harder.

First, observe that $\hat{E}$ is really just the random variable that counts the number of heads in n tosses, and so has a binomial distribution with a probability of success on each toss of p. It follows that $E\left(\hat{E}\right) = np$ and $V\left(\hat{E}\right) = npq = np(1-p)$.

Second, since $\hat{p} = \frac{\hat{E}}{n}$ it follows that $E\left(\hat{p}\right) = E\left(\frac{\hat{E}}{n}\right) = \frac{1}{n}E\left(\hat{E}\right) = \frac{1}{n} \cdot np = p$, and that $V\left(\hat{p}\right) = .V\left(\frac{\hat{E}}{n}\right) = \frac{1}{n^2}V\left(\hat{E}\right) = \frac{1}{n^2} \cdot np(1-p) = \frac{p(1-p)}{n}$. It also follows that the standard deviation of $\hat{p}$ is $\sigma = \sqrt{V\left(\hat{p}\right)} = \sqrt{\frac{p(1-p)}{n}}$.

Third, observe that $P(0.95p \leq \hat{p} \leq 1.05p) \geq P(0.95p < \hat{p} < 1.05p)$. If we can get the latter probability to be at least 0.95, then the former one will be too.

Fourth, cconsider the following rearrangement of the probability, $P(0.95p < \hat{p} < 1.05p)$, that we are now trying to get be at least 0.95.

$$\begin{aligned}
P(0.95p < \hat{p} < 1.05p) &= P(0.95p - p < \hat{p} - p < 1.05p - p) \\
&= P(-0.05p < \hat{p} - p < 0.05p) \\
&= P\left(|\hat{p} - p| < 0.05p\right) \\
&\geq P\left(|\hat{p} - p| < 0.05 \cdot 0.4\right) = P\left(|\hat{p} - p| < 0.02\right)
\end{aligned}$$

The inequality in this chain follows because we know that $p \leq 0.4$, and if we take the probability over smaller set of outcomes, we get a smaller probability. Thus, if we can get $P\left(|\hat{p}-p|<0.02\right)$ to be at least 0.95, $P(0.95p < \hat{p} < 1.05p)$, and hence $P(0.95p \leq \hat{p} \leq 1.05p)$, will be at least 0.95 too.

Fifth, observe that $P\left(|\hat{p}-p|<0.02\right) = 1 - P\left(|\hat{p}-p| \geq 0.02\right)$. It follows that we will have $P\left(|\hat{p}-p|<0.02\right) \geq 0.95$ exactly when $P\left(|\hat{p}-p| \geq 0.02\right) \leq 0.05$.

Sixth, since $E\left(\hat{p}\right) = p$, we can apply Chebyshev's Inequality to get an upper bound for $P\left(|\hat{p}-p| \geq 0.02\right)$. If we get that upper bound, in turn, to be at most 0.05, we will have that $P\left(|\hat{p}-p| \geq 0.02\right) \leq 0.05$, and hence that $P(0.95p \leq \hat{p} \leq 1.05p) \geq 0.95$, as desired.

Seventh, to apply Chebyshev's Inequality, we need to decompose 0.02 as $k\sigma$, where σ is the standard deviation of $\hat{p}$. Since $\hat{p}$ has standard deviation $\sigma = \sqrt{\frac{p(1-p)}{n}}$, $0.02 = k\sigma = k\sqrt{\frac{p(1-p)}{n}}$, so $k = \frac{0.02}{\sqrt{\frac{p(1-p)}{n}}} = \frac{0.02\cdot\sqrt{n}}{\sqrt{p(1-p)}}$.

Eighth, we apply Chebyshev's Inequality:

$$\begin{aligned} P\left(|\hat{p}-p| \geq 0.02\right) &= P\left(|\hat{p}-p| \geq k\sigma\right) \\ &\leq \frac{1}{k^2} = \frac{1}{\left(\frac{0.02\cdot\sqrt{n}}{\sqrt{p(1-p)}}\right)^2} = \frac{p(1-p)}{0.0004\cdot n} \end{aligned}$$

Thus, if get $\frac{p(1-p)}{0.0004\cdot n} \leq 0.05$, we will have that $P\left(|\hat{p}-p| \geq 0.02\right) \leq 0.05$, and hence that $P(0.95p \leq \hat{p} \leq 1.05p) \geq 0.95$, as desired.

Ninth, we observe that $p(1-p) \leq \frac{1}{4} = 0.25$ no matter what p is. (You can use calculus to check that the function $f(x) = x(1-x) = x - x^2$ has a global maximum at $x = \frac{1}{2}$, at which point its value is $\frac{1}{4}$.) It follows that:

$$P\left(|\hat{p}-p| \geq 0.02\right) \leq \frac{p(1-p)}{0.0004\cdot n} \leq \frac{\frac{1}{2}}{0.0004\cdot n} = \frac{1}{0.0008\cdot n}$$

Tenth, observe that $\frac{1}{0.0008\cdot n} \leq 0.05$ exactly when $1 \leq 0.05\cdot 0.0008\cdot n = 0.00004n$, which, in turn, happens exactly when $n = \frac{1}{0.00004} = 25000$. Thus, if we toss the coin $n = 25000$ times, we are guaranteed that $P\left(|\hat{p}-p| \geq 0.02\right) \leq 0.05$, which guarantees that $P\left(|\hat{p}-p| < 0.02\right) \geq 0.95$, which guarantees that $P(0.95p < \hat{p} < 1.05p) \geq 0.95$, as desired.

Tossing the coin $n = 25000$ times is probably very serious overkill to ensure that $P(0.95p < \hat{p} < 1.05p) \geq 0.95$, but it might be less painful than trying to find the minimum n required ... $\square$