Mathematics 1550H – Probability I: Introduction to Probability

TRENT UNIVERSITY, Summer 2023 (S62)

Assignment #5

Chebyshev's Inequality

Due just before midnight on Friday, 21 July.*

Suppose the random variable Y counts the number of heads in 10 tosses of a biased coin with $P(H) = 0.6$ and $P(T) = 0.4$ on each toss. Y then has a binomial distribution: it can take on the integer values 0 through 10, each with with probability $P(Y = y) = \binom{10}{y} \cdot 0.6^y \cdot 0.4^{10-y}$. It also has expected value $E(X) = 10 \cdot 0.6 = 6$ and variance $V(X) = 10 \cdot 0.6 \cdot 0.4 = 2.4$.

1. Work out $P\left(|Y - 6| > 3\right)$ as precisely as you can. *[3]*

Chebyshev's Inequality, which we'll see in class later on, lets us estimate probabilities involving random variables knowing nothing about the distribution involved except for its expected value and standard deviation. Mind you, these estimates are usually pretty crude, but they are pretty easy to get.

There are several ways to state this inequality; one common version states that a random variable X, which could be discrete or continuous, with expected value $\mu = E(X)$ and standard deviation $\sigma = \sqrt{V(X)}$ must satisfy

$$P\left(|X - \mu| \geq k\sigma\right) \leq \frac{1}{k^2}$$

for any real number $k > 0$. Technically, this form of Chebyshev's Inequality gives us upper bounds for certain probabilities.

If the mean and/or the standard deviation are not defined or are unknown for X, Chebyshev's Inequality isn't useful. Note also that it is of interest only if $k > 1$, since if $k \leq 1$, then $\frac{1}{k^2} \geq 1$ and any probability must be ≤ 1.

2. Use Chebyshev's Inequality to find an upper bound for the probability $P\left(|Y - 6| > 3\right)$ in question **1**. How close is this upper bound to the precise value? *[3]*

3. Either find

i. a probability density function $f(x)$ for a continuous random variable such that a continuous random variable X with this density will have $\mu = 0$, $\sigma = 1$, and satisfy $P\left(|X| \geq 2\right) = \frac{1}{4}$, or

ii. a probability distribution function $m(x)$ for a discrete random variable X such that a discrete random variable X with this distribution function will have $\mu = 0$, $\sigma = 1$, and satisfy $P\left(|X| \geq 2\right) = \frac{1}{4}$. *[4]*

NOTE: The point of **3** is that Chebyshev's Inequality can't really be improved upon, short of having additional information about the distribution of the random variable X.

* You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If this fails, you may submit your work to the instructor on paper or by email to `sbilaniuk@ trentu.ca` .