Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2023 (S62)

Solutions to Assignment #4
A Continuous Grind

Suppose X is a continuous random variable whose probability density function is $f(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$.

1. Verify that $f(x)$ is indeed a valid probability density function. *[1.5]*

SOLUTION. *[We did this in class the other week ...]* We check the two conditions a valid probability density function needs to satisfy.

First, we have that $f(x) = 0 \geq 0$ when $x < 0$ and $f(x) = xe^{-x} \geq 0$ when $x \geq 0$ (since then $x \geq 0$ and $e^{-x} > 0$), so $f(x) \geq 0$ for all x, as required.

Second, we will compute the necessary integral with the help of the substitution $w = -x$, so $x = (-1)w$ and $dx = (-1)\,dw$, changing the limits as we go along: $\begin{matrix} x & 0 & \infty \\ w & 0 & -\infty \end{matrix}$ Then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} xe^{-x}\,dx = 0 + \int_{0}^{-\infty} (-1)we^{w}(-1)\,dw = \int_{0}^{-\infty} we^{w}\,dw$$

We will now apply integration by parts, with $u = w$ and $v' = e^w$, so $u' = 1$ and $v = e^w$.

$$= we^w\Big|_0^{-\infty} - \int_0^{-\infty} 1e^w\,dw = we^w\Big|_0^{-\infty} - e^w\Big|_0^{-\infty}$$
$$= \left[-\infty e^{-\infty} - 0e^0\right] - \left[e^{-\infty} - e^0\right] = \left[-\frac{\infty}{e^{\infty}} - 0\right] - [0 - 1]$$
$$= [0 - 0] - [0 - 1] = 1\,, \text{ as required.}$$

Since both of the necessary conditions for being a valid probability density function are satisfied, $f(x)$ really is one. □

2. Compute the expected value $E(X)$ of X. *[1.5]*

SOLUTION. We'll use the same substitution used above. The expected value of a random variable X with density function $f(x)$ is, by definition:

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\infty} x \cdot xe^{-x}\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} x^2e^{-x}\,dx$$
$$= 0 + \int_0^{\infty} x^2e^{-x}\,dx = \int_0 -\infty(-w)^2e^w(-1)\,dw = \int_{-\infty}^{0} w^2e^w\,dw$$

Integration by parts again, with $u = w^2$ and $v' = e^w$, so $u' = 2w$ and $v = e^w$.

$$= w^2e^w\Big|_{-\infty}^{0} - \int_{-\infty}^{0} 2we^w\,dw = \left[0^2e^0 - (-\infty)^2e^{-\infty}\right] - 2\int_{-\infty}^{0} we^w\,dw$$
$$= \left[0 - \frac{\infty^2}{e^{\infty}}\right] - 2(-1)\int_0^{-\infty} we^w\,dw = [0 - 0] + 2\int_0^{-\infty} we^w\,dw = 2 \cdot 1 = 2$$

Note that we know from the solution to question **1** that $\displaystyle\int_0^{-\infty} we^w\,dw = 1$, so there was no need to work it out again.

Thus the expected value of the random variable X is $E(X) = 2$. □

3. Compute the variance $V(X) = E\left(X^2\right) - [E(X)]^2$ of X. *[2]*

SOLUTION. We first need to compute $E\left(X^2\right)$; we'll use the same substitution used above one more time.

$$\begin{aligned}
E\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{0} x^2\cdot 0\,dx + \int_0^{\infty} x^2\cdot xe^{-x}\,dx = \int_{-\infty}^{0} 0\,dx + \int_0^{\infty} x^3e^{-x}\,dx \\
&= 0 + \int_0^{\infty} x^3e^{-x}\,dx = \int_0^{-\infty} (-w)^3e^w(-1)\,dw = \int_0^{-\infty} w^3e^w\,dw
\end{aligned}$$

Integration by parts, now with $u = w^3$ and $v' = e^w$, so $u' = 3w^2$ and $v = e^w$.

$$\begin{aligned}
&= w^3e^w\Big|_0^{-\infty} - \int_0^{-\infty} 3w^2e^w\,dw = \left[(-\infty)^3e^{-\infty} - 0^3e^0\right] - 3\int_0^{-\infty} w^2e^w\,dw \\
&= \left[-\frac{\infty^3}{e^\infty} - 0\right] - 3\cdot(-1)\int_{-\infty}^{0} w^2e^w\,dw = [0-0] + 3\int_{-\infty}^{0} w^2e^w\,dw = 3\cdot 2 = 6
\end{aligned}$$

Note that we know from the solution to question **2** that $\displaystyle\int_{-\infty}^{0} w^2e^w\,dw = 2$, so there was no need to work it out again.

Finally, we also know from **2** that $E(X) = 2$. Thus the variance of the random variable X is $V(X) = E\left(X^2\right) - [E(X)]^2 = 6 - 2^2 = 6 - 4 = 2$. □