Mathematics 1550H – Probability I: Introduction to Probability
TRENT UNIVERSITY, Summer 2023 (S62)

Solutions to Assignment #2
A Hand of Ill-Omen

Recall that a standard 52-card deck has 4 *suits*, namely $\heartsuit$, $\diamondsuit$, $\clubsuit$, and $\spadesuit$, each of which has 13 cards of different *kinds* or *ranks*, namely A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, and 2.

One infamous five-card hand is the *Dead Man's Hand*, consisting of $A\clubsuit$, $A\spadesuit$, $8\clubsuit$, $8\spadesuit$, and $Q\heartsuit$. According to legend – there seem to be no contemporary accounts that describe the hand – this is the poker hand that James Butler "Wild Bill" Hickok held when he was shot in the back of the head in a saloon in the town of Deadwood in the then Dakota Territory in 1876.

1. If you draw a five-card hand at random and all at once (so order doesn't matter), what is the probability that it will be the Dead Man's Hand? *[1]*

SOLUTION. There are $\binom{52}{5}$ possible five-card hands if order doesn't matter, each equally likely in a completely random draw. It follows that the probability that any particular one, such as the Dead Man's Hand, is drawn is:

$$\frac{1}{\binom{52}{5}} = \frac{1}{\frac{52!}{(52-5)!5!}} = \frac{47!5!}{52!} = \frac{120}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{120}{311875200} = \frac{1}{2598960} \approx 0.0000003848$$

Obviously, the probability of any particular hand being drawn is pretty low. $\square$

2. If you draw a five-card hand at random and all at once (so order doesn't matter), what is the probability that it will have no card in common with the Dead Man's Hand? *[2]*

SOLUTION. There are $52 - 5 = 47$ cards in a standard 52-card deck that are not in the Dead Man's Hand, so there are $\binom{47}{5}$ possible five-card hands that have no card in common with the Dead Man's Hand. Since every hand is as likely as any other, it follows that the probability of drawing a five-card hand that has no cards in common with the Dead Man's Hand is:

$$\frac{\binom{47}{5}}{\binom{52}{5}} = \frac{\frac{47!}{(47-5)!5!}}{\frac{52!}{(52-5)!5!}} = \frac{47!}{42!5!} \cdot \frac{47!5!}{52!} = \frac{47 \cdot 46 \cdot 45 \cdot 44 \cdot 43}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{184072680}{311875200} = \frac{511313}{866320} \approx 0.5902$$

Thus the probability of drawing a hand that has no card in common with the Dead Man's Hand is decent, but not overwhelmingly good. $\square$

3. If you draw a five-card hand at random and all at once (so order doesn't matter), what is the probability that it will have exactly one card in common with the Dead Man's Hand? *[2]*

SOLUTION. Consider a five-card hand with exactly one card in common with the Dead Man's Hand. There are $\binom{5}{1} = 5$ ways to choose which card from the Dead Man's Hand is in the hand, and $\binom{47}{4}$ ways to pick the remaining four cards from the 47 cards in the deck which are not in the Dead Man's Hand. Since every hand is as likely as any other, it follows that the probability of drawing a five-card hand that has exactly one card in common with the Dead Man's Hand is:

$$\frac{\binom{5}{1}\binom{47}{4}}{\binom{52}{5}} = \frac{5 \cdot \frac{47!}{(47-4)!4!}}{\frac{52!}{(52-5)!5!}} = 5 \cdot \frac{47!}{43!4!} \cdot \frac{47!5!}{52!} = \frac{5 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 5}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}$$
$$= \frac{107019000}{311875200} = \frac{59455}{173264} \approx 0.3431$$

Not as high as the probability drawing a hand that has no card in common with the Dead Man's Hand, but nothing to sneeze at. □

4. If you draw a five-card hand at random and all at once (so order doesn't matter), what is the probability that it will have at least one card in common with the Dead Man's Hand? *[5]*

Solution. The probability that a randomly drawn five-card hand will have at least one card in common with the Dead Man's Hand is 1 minus the probability that it has no card card in common with the Dead Man's Hand, which we know from **2**. Thus:

$$1 - \frac{\binom{47}{5}}{\binom{52}{5}} = 1 - \frac{511313}{866320} = \frac{355007}{866320} \approx 0.4098$$

When I was making up this assignment, I somehow hallucinated that this was hard. My apologies to those I told that, some of whom had suggested the above solution. It didn't penetrate for me until the evening of Tuesday, 26 June, when typing up these solutions. □